Mathematical Reviews

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ALGEBRA

Schöbe, Waldemar. Das Lucassche Ehepaarproblem. Math. Z. 48, 781–784 (1943). [MF 9014]

The Lucas problem referred to is the following one on discordant permutations: n wives are seated at equal intervals around a circular table with a seating capacity of $2n$; to find a formula for the number A_n of different ways that their husbands can be seated provided no husband is seated next to his own wife ($A_4=2$, $A_5=13$, $A_6=80$, $\cdots$). Moreau has shown [Lucas, Théorie des Nombres, Gauthier-Villars, Paris, 1891, pp. 491–495] that, if we introduce b_n by

$$(1) \qquad nb_n=A_n-2(-1)^n,$$

then the b's are integers satisfying the difference equation

$$b_{n+1}=nb_n+b_{n-1}+2(-1)^n,$$

from which he calculated the first 20 A's. The author's contribution to the problem is to obtain a solution of this difference equation in terms of "sub-factorial n." If we denote this by

$$h_n=n!\sum_{\nu=0}^{n}(-1)^\nu/\nu!,$$

then it is shown that

$$(2) \qquad n!b_{n+1}=(-1)^n\sum_{\lambda=0}^{n}(-1)^\lambda\binom{n}{\lambda}h_\lambda^2,$$

whence by (1) a formula for A_n follows at once. A probability argument can be used to support the conjecture that $A_n\sim e^{-2}n!$. A rigorous proof of this fact, or, what is the same thing, that $b_{n+1}\sim e^{-2}n!$, follows at once from (2) using the fact that $h_\lambda=e^{-1}\lambda!+O(\lambda^{-1})$. *D. H. Lehmer.*

Bhattacharya, K. N. A note on two-fold triple systems. Sankhyā 6, 313–314 (1943). [MF 9079]

For the existence of a two-fold triple system of v different symbols arranged in b triplets in such a way that every symbol is replicated r times, it has been known to be necessary that $vr=3b$, $v=r+1$. R. C. Bose [Ann. Eugenics 9, 353–399 (1939); these Rev. 1, 199] has shown that two-fold triple systems for every v, b and r satisfying these conditions can be obtained if two auxiliary problems can be solved. These auxiliary problems consist in constructing for every t, $2t+1$ and $2t$ triplets of integers satisfying certain conditions. The author gives an explicit solution for these auxiliary problems for every value of t. Together with Bose's method his result furnishes a systematic method for the construction of two-fold triple systems for every value of v, b and r satisfying the conditions $vr=3b$, $v=r+1$.
H. B. Mann (Barrytown, N. Y.).

Nair, K. R. Certain inequality relationships among the combinatorial parameters of incomplete block designs. Sankhyā 6, 255–259 (1943). [MF 9078]

R. A. Fisher [Ann. Eugenics 10, 52–75 (1940); these Rev. 1, 348] has shown that in every balanced incomplete block design $b\geqq v$ (b=number of blocks, v=number of varieties). For resolvable balanced incomplete block designs R. C. Bose [Sankhyā 6, 105–110 (1942); these Rev. 4, 237] derived the inequality $b\geqq v+r-1$, where r is the number of replications. For partially balanced incomplete block designs these inequalities are not necessarily true. For these designs the author derives a sufficient condition in determinantal form for the inequalities $b\geqq v$ in the general case and $b\geqq v+r-1$ for resolvable designs. He shows that for balanced incomplete block designs this condition is always fulfilled. Thus his result contains the results of R. A. Fisher and R. C. Bose. For balanced incomplete block designs he derives the sharper inequalities

$$b\geqq 1+k(r-1)^2/[(r-k)+\lambda(k-1)]$$

in the general case and

$$b\geqq rk(r-1)/[(r-k)+\lambda(k-1)]$$

for resolvable designs. *H. B. Mann.*

Vijayaraghavan, T. On symmetric polynomial functions of zeros of polynomials. Math. Student 10, 113–114 (1942). [MF 8740]

Lipka, Stephan. Über die Vorzeichenregeln von Budan-Fourier und Descartes. Jber. Deutsch. Math. Verein. 52, 204–217 (1942). [MF 9054]

Let $f(x)$ be a real polynomial of degree n, let $V(x)$ be the number of variations of sign in the sequence $f(x), f'(x), \cdots, f^{(n)}(x)$ and let $f(a)f(b)\neq 0$ with $a<b$. According to the well-known theorem of Budan-Fourier, the difference $\Delta=V(a)-V(b)$, if not equal to the number N of zeros of $f(x)$ in the interval (a, b), exceeds this number by an even integer. In the paper under review, Lipka establishes as sufficient condition for $\Delta=N$ that the interval (a, b) lie outside all the Jensen circles of $f(x)$ and its first $n-2$ derivatives. (By the Jensen circles of a polynomial is meant the circles which have as diameters the line-segments joining pairs of its conjugate complex zeros.) Lipka then applies his theorem to Descartes' rule. As sufficient condition for the number of positive zeros of $f(x)$ to equal the number of variations of sign in the coefficients of $f(x)$, he requires that all the non-real zeros of $f(x)$ lie in the circle $|x+d|=r$ with d real and $d>r\sqrt{n}$, or that they all lie in the angular opening $|\arg(-x)|<\tan^{-1}(n^{-\frac{1}{2}})$. *M. Marden.*

Bhalotra, Y. and Chowla, S. Some theorems concerning quintics insoluble by radicals. Math. Student 10, 110–112 (1942). [MF 8739]

This note contains elementary proofs of results such as the following. The equation $x^5+cx+d=0$, where c and d are integers, if irreducible, is not solvable by radicals if c is odd and not divisible by any prime equivalent to 3 (mod 4). *R. D. James* (Vancouver, B. C.).

Mann, H. B. Quadratic forms with linear constraints. Amer. Math. Monthly **50**, 430–433 (1943). [MF 9092]

The author obtains, in a new form, necessary and sufficient conditions that a quadratic form $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ be positive (negative) for all values of the unknowns satisfying the equations $\sum_{j=1}^{n} b_{kj}x_j=0$, $k=1, 2, \cdots, p<n$.

N. H. McCoy (Northampton, Mass.).

Oldenburger, Rufus. The characteristic of a quadratic form for an arbitrary field. Trans. Amer. Math. Soc. **53**, 454–462 (1943). [MF 8419]

The coefficient field K is of characteristic not equal to 2. The characteristic of a quadratic form Q over K is defined as the maximum number σ of linearly independent linear forms $L_1, \cdots, L_\sigma$ such that the rank of $Q+\lambda_1L_1^2+\cdots+\lambda_\sigma L_\sigma^2$ is equal to the rank of Q for all values of the λ's. The number σ is shown to be identical with the characteristic of Q as defined by Witt and, if K is real, with that of Loewy. Under addition of a term λL^2, L linear, to a quadratic form Q, the characteristic σ of Q changes by at most 1.

C. C. MacDuffee (Madison, Wis.).

Parker, W. V. Limits to the characteristic roots of a matrix. Duke Math. J. **10**, 479–482 (1943). [MF 8969]

Let $A=(a_{ij})$ be a square matrix with real or complex elements and set $R_i=\sum_{j=1}^{n}|a_{ij}|$, $T_j=\sum_{i=1}^{n}|a_{ij}|$. Then any characteristic root λ of A does not exceed in absolute value the least of the R_i, T_j. A similar lower bound for $|\lambda|$ is obtained as are also bounds for the real and imaginary parts of the characteristic roots.

N. H. McCoy.

Collatz, L. Einschliessungssatz für die charakteristischen Zahlen von Matrizen. Math. Z. **48**, 221–226 (1942). [MF 8549]

Let A be a real square matrix, $U=\{u_1, \cdots, u_n\}$ a column vector, and define $V=\{v_1, \cdots, v_n\}$ by $V=AU$. It is shown that, if A is symmetric, the interval between the minimum and the maximum of the ratios v_i/u_i contains at least one characteristic value of A. A similar theorem holds also if $a_{ik}>0$ and $u_i>0$.

W. Feller (Providence, R. I.).

Yen, Chih-ta. On matrices whose associated matrices are equal. Acad. Sinica Science Record **1**, 87–90 (1942). [MF 8844]

The author proves the following theorem. "Let $T(A)$ be a nonsingular irreducible associated matrix of a matrix A of order n. If the index r of $T(A)$ is less than n and if one of the matrices A or B is nonsingular, then $T(A)=T(B)$ if and only if $A=\omega B$, where ω is an rth root of unity." [The particular case of this theorem when $T(A)$ is the rth compound of A had been proved previously [Williamson, Bull. Amer. Math. Soc. **39**, 108–111 (1933)].] The author first notes that his theorem is equivalent to the theorem that $T(A)$ is the identity matrix if and only if $A=\omega$. In proving this he shows that each characteristic number of $T(A)$ must have the value ω and then by a reduction of A to Jordan canonical form that each elementary divisor of $T(A)-x$ must be linear. His methods and notation are based on Wedderburn's exposition of associated matrices [J. H. M. Wedderburn, Lectures on Matrices, Amer. Math. Soc. Colloquium Publ., v. 17, New York, 1934, pp. 76–87].

J. Williamson (Flushing, N. Y.).

This paper is identical with the one in p. 225.

Eckmann, Beno. Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen. Comment. Math. Helv. **15**, 358–366 (1943).

This is a simplified proof using representation theory and group characters of the following theorem of A. Hurwitz and J. Radon. Let n be written $n=u\cdot 2^{4\alpha+\beta}$ (u odd; $\beta=0, 1, 2, 3$). There exist n complex bilinear forms $z_1, \cdots, z_n$ in the two sets of variables $x_1, \cdots, x_p$ and $y_1, \cdots, y_n$ such that

$$(x_1^2+\cdots+x_p^2)(y_1^2+\cdots+y_n^2)=z_1^2+\cdots+z_n^2$$

if and only if $p\leqq 8\alpha+2^\beta$. The forms z_i may always be chosen to be real.

C. C. MacDuffee (Madison, Wis.).

Wold, Herman. On infinite, non-negative definite, Hermitian matrices, and corresponding linear equation systems. Ark. Mat. Astr. Fys. **29**, 13 pp. (1943). [MF 8785]

Let $H=\|h_{ij}\|$ be an infinite, positive definite, Hermitian matrix. The author considers the infinite system of linear equations represented in vector notation by $Hx=c$. In terms of H he defines two infinite matrices G and S such that $g_{ij}=0$ for $j>i$ and shows that all solutions of (1) $Hx=0$ satisfying certain conditions must also be solutions of the system (2) $Gx=0$; $Sx=0$. In particular, a vector x satisfying

$$\sum_{1}^{\infty}\sqrt{h_{ii}}\,|x_i|<\infty \tag{3}$$

is a solution of (1) if and only if it is a solution of (2). If the matrix S vanishes (as it will, for instance, if there are only a finite number of nonzero elements in each row of H) or is of finite rank, the general solution of (2) may be obtained by an infinite sequence of finite calculations. Similar results hold in the nonhomogeneous case. One illustration of the method is given; detailed proofs, generalizations and other illustrations are to be published later.

D. Blackwell.

Perron, Oskar. Über eine für die Invariantentheorie wichtige Funktionalgleichung. Math. Z. **48**, 136–172 (1942). [MF 8566]

The author considers the matrix equation $\Phi(X)\Phi(Y)=\Phi(XY)$, where X and Y are matrices of order n with elements which are complex numbers and where $\Phi(X)$ is a matrix of order ν whose elements are functions of the elements of X. He is interested in solutions analytic in the neighborhood of E, the unit matrix of order n, and notes the obvious solution $\Phi(X)=|X|^A$, where A is any matrix whose elements are constants. For $\nu=1, 2, 3$ he obtains in addition all other solutions. The author seems to be unaware of I. Schur's work on invariant matrices [Dissertation, Berlin, 1901].

J. Williamson (Flushing, N. Y.).

***MacDuffee, Cyrus Colton. Vectors and Matrices.** Carus Monograph Series, no. 7. Mathematical Association of America, Ithaca, N. Y., 1943. xi+192 pp. \$2.00.

This is an exposition, from the vector point of view, of certain fundamental topics in the theory of matrices with elements in a field. The subject of central interest is the rational reduction of a matrix to canonical form, and the sequence of topics discussed leads naturally and easily to this goal. There are, however, a considerable number of by-products which are of interest in themselves. A brief summary by chapters follows. I. Systems of linear equations. Solutions are found by making use of elementary operations upon the equations of a system to yield an

equivalent system of simple type. II. Vector spaces. Basis and rank of a vector space; matrix as generalization of the concept of vector; elementary matrices; Hermite canonical (triangular) form of a matrix; systems of linear equations in matrix notation; rank of a product. III. Determinants. The determinant of matrix X is introduced as a number $d(X)$ such that $d(X)$ is of lowest possible degree >0 in the elements of X considered as independent indeterminates and such that $d(XY)=d(X)d(Y)$. Adjoint; Cramer's rule; minors and co-factors; rank. IV. Matric polynomials. Ring with unit element; polynomial domains; characteristic and minimum functions; rank of a matrix polynomial; matrix having a prescribed minimum function. V. Union and intersection. Complementary vector spaces; union and intersection of vector spaces; greatest common right (left) divisor and least common left (right) multiple of matrices and the relation to the spaces spanned by row (column) vectors; sum of vector spaces; annihilators of vectors. VI. The rational canonical form. The reduction is accomplished easily by making use of developments of preceding chapters. Both the derogatory and non-derogatory cases are treated in detail. VII. Elementary divisors. Equivalence of matrices; invariant factors and elementary divisors; Segre and Weyr characteristics; collineations. VIII. Orthogonal transformations. Orthogonal matrices and orthogonal bases; orthogonal canonical form of a real symmetric matrix; principal axis transformation. IX. Endomorphisms. An introduction to the concept of matrix as an endomorphism of a finite vector space. There is included a bibliography on the rational canonical form. *N. H. McCoy.*

Abstract Algebra

Levi, F. W. Modern algebra. Bull. Calcutta Math. Soc. **35**, 1–6 (1943). [MF 9447]

Presidential Address.

Birkhoff, Garrett. What is a lattice? Amer. Math. Monthly **50**, 484–487 (1943). [MF 9259]

Expository article.

Campbell, Alan D. Set-coordinates for lattices. Bull. Amer. Math. Soc. **49**, 395–398 (1943). [MF 8386]

The author defines a one-one correspondence $a\leftrightarrow A$ of the elements of a lattice L by sets to be a representation of L if it preserves meets. It is known that every lattice has one representation. He obtains a representation of the free modular lattice with three generators by sets having as few ($\leqq 9$) points as possible. He develops a notation for the representation of lattices by sets, and expresses the modular and distributive laws in terms of this notation.

G. Birkhoff (Cambridge, Mass.).

Klein-Barmen, Fritz. Über gewisse Halbverbände und kommutative Semigruppen. I. Math. Z. **48**, 275–288 (1942). [MF 8554]

The author defines postulationally two very general types of algebra with binary operation (multiplication): holoids and associatives. He develops the ideas of identity, zero, divisor, coprimeness, prime with great generality, by selecting the weakest postulates needed to prove their most elementary properties. *G. Birkhoff* (Cambridge, Mass.).

Klein-Barmen, Fritz. Über gewisse Halbverbände und kommutative Semigruppen. II. Math. Z. **48**, 715–734 (1943). [MF 9010]

[Cf. the preceding review.] The author defines a holoid to be a commutative, associative, multiplication system with unity, in which associate elements are identified ($a=bx$, $b=ay$ implies $a=b$). Lattices are special cases. Any holoid can be partially ordered by the relation a divides b. It is shown how, in the linearly ordered case, the location of idempotents determines the multiplication rule for certain other elements. *G. Birkhoff* (Cambridge, Mass.).

✶Jacobson, Nathan. The Theory of Rings. American Mathematical Society Mathematical Surveys, vol. I. American Mathematical Society, New York, 1943. vi +150 pp. $2.25.

The book is mainly concerned with the theory of rings in which both maximal and minimal conditions hold for ideals (except in the last chapter, where rings of the type of a maximal order in an algebra are considered). The contents therefore cover less ground than the title would seem to indicate, since the theory of commutative rings with maximal condition only is entirely omitted. The central idea consists in representing rings as rings of endomorphisms of an additive group, which can be achieved by means of the regular representation.

Chapters I and II establish the fundamental facts about groups with operators and vector spaces (over division rings). Chapter III, which is concerned with the theory of principal ideal rings, is more or less a digression from the general theme of the book. The existence and invariance of elementary divisors for a matrix with coefficients in a principal ideal ring are proved. The whole theory is applied to the study of a semi-linear transformation, including the proof of a generalized "Hauptgeschlechtsatz" [theorem 27, p. 47]. Chapter IV contains the structure theory of rings of endomorphisms and its applications to the structure theory of abstract rings. The classical Wedderburn-Artin structure theorems are proved for rings satisfying the descending chain condition; the proof follows the pattern of the improvement made by R. Brauer on the Hopkins method. Applications are made to the theory of projective representations of a finite group and of crossed products algebras defined by means of a factor set. Chapter IV ends with the "Galois theory" of a division ring.

The special properties of those rings which are algebras over a field are studied in chapter V. There is not much to add to the results of chapter IV as far as structure theory is concerned, but the notion of direct product of algebras brings with itself a new series of notions and results: Brauer group, representations of an algebra in a division algebra, crossed products. The chapter ends with a discussion of the minimal polynomial of an element in an algebra and the theorem of Wedderburn on the existence of a semi-simple subalgebra which gives a complete set of representatives for the residue classes modulo the radical. Chapter V treats of the ideal theory in a maximal order of a ring satisfying both chain conditions: factorization theorem for two sided ideals, Brandt groupoid and factorization of one sided normal ideals.

On the whole, the book is an up to date and very clear exposition of a large portion of modern algebra. The general idea of using representation theory (in a generalized sense) in order to obtain the theorems of structure is followed in a very consequent way throughout the book. The book is

quite accessible to a beginner, and provides at each step interesting applications of the general theory. The only criticism of the reviewer would be directed at the omission of the theory of symmetric and related algebras.

C. Chevalley (Princeton, N. J.).

Harrison, Gerald. The structure of algebraic moduls. Proc. Nat. Acad. Sci. U. S. A. **28**, 410–413 (1942). [MF 7286]

Let $(\omega_1, \omega_2, \cdots, \omega_n)$ denote the modul (called the unit modul) with basis elements $\omega_1, \omega_2, \cdots, \omega_n$ linearly independent and each of infinite order with respect to the domain of rational integers. Beginning with the unit modul and forming all possible successive coverings with respect to a fixed prime p there is obtained a sublattice of the lattice of all submoduls of the unit modul. Such a lattice is called a p-lattice. Let $p_1, p_2, \cdots, p_i, \cdots$ denote the set of all rational integral primes; to each there corresponds a p_i-lattice and, if M is any submodul of the unit modul, then

$$M = M_{p_1} \cap M_{p_2} \cap \cdots \cap M_{p_i} \cap \cdots,$$

where M_{p_i} is that modul of the p_i-lattice which is the meet of all moduls belonging to the p_i-lattice and containing M. This representation is unique. The lattice of all submoduls of the unit modul is the direct product of the p_i-lattices. If the unit modul has the form $(1, \omega)$, where ω is a quadratic algebraic integer, and if M and N are submoduls, then in terms of their respective representations described above

$$MN = M_{p_1}N_{p_1} \cap M_{p_2}N_{p_2} \cap \cdots \cap M_{p_i}N_{p_i} \cap \cdots.$$

(It is conjectured that this property holds for ω of nth degree.) The following corollaries are noted. If in each of two moduls one of the components with respect to each prime is the unit modul, the product of the moduls is an ideal. A submodul of $(1, \omega)$ is a ring (or an ideal) if, and only if, each of its components is. The further study of the multiplicative and lattice theoretic decomposition of moduls may be restricted to their p_i-lattice components.

J. L. Dorroh (Baton Rouge, La.).

Johnson, R. E. On structures of infinite modules. Trans. Amer. Math. Soc. **53**, 469–489 (1943). [MF 8421]

This is the development of a structure theory for certain infinite modules with countable bases, with application to infinite matrices. For a commutative field P, Ξ is assumed to be a universal P-module with a countable P-basis. A principal ideal ring Q containing P is considered as an operator domain of Ξ. Then the main problem is to determine under what conditions submodules of Ξ have proper Q-bases. First, a complete characterization is given by means of an infinite matrix for the proper Q-bases of any Q-submodule of Ξ. In the finite case it is shown that every Q-module with a finite Q-basis has a proper Q-basis. The concepts of primitivity (Chevalley) and index are important in determining conditions for a Q-module to have a proper Q-basis. To find these conditions, the nonregular elements H of Ξ are split from Ξ. The resulting Q-module Ξ/H is regular. Necessary and sufficient conditions are found for both H and Ξ/H to have proper Q-bases. If $Q/(m)$, m not a unit of Q, is taken as the operator domain of Ξ, then Ξ possesses a proper $Q/(m)$-basis. If Ξ is taken to be the set of all vectors over P of order type ω which are finitely nonzero, the total operator domain of Ξ is a certain ring $\mathfrak{M}_\omega$ of infinite matrices. Then an element A of $\mathfrak{M}_\omega$ can be transformed into a direct sum of finite matrices only if Ξ has a proper $P[A]$-basis.

C. C. MacDuffee.

Levitzki, Jakob. Semi-nilpotent ideals. Duke Math. J. **10**, 553–556 (1943). [MF 8979]

A ring is semi-nilpotent if every subring generated by a finite number of elements is nilpotent. In a previous paper [Bull. Amer. Math. Soc. **49**, 462–466 (1943); these Rev. **4**, 238] the author proposed as the definition of the radical of a ring the union of its semi-nilpotent ideals; in the present paper he considers the nilpotency of this radical. In a similar investigation [Bull. Amer. Math. Soc. **48**, 752–758 (1942); these Rev. **4**, 70] R. Brauer showed, on the assumption of the minimal condition for two-sided nil-ideals, that the union of all nilpotent ideals is nilpotent. The author generalizes Brauer's result, both in hypothesis and conclusion, by proving that the minimal condition for two-sided semi-nilpotent ideals implies that his radical is nilpotent. Some immediate consequences are derived concerning the semi-simplicity of a ring modulo its radical.

I. Kaplansky (Cambridge, Mass.).

Dieudonné, J. Sur les systèmes hypercomplexes. J. Reine Angew. Math. **184**, 178–192 (1942). [MF 9037]

This paper develops the proofs of the results announced by the author in a note in the C. R. Acad. Sci. Paris **211**, 172–174 (1940)[these Rev. **3**, 101]. Let $\mathfrak{o}$ be a ring which satisfies the minimal conditions for both left and right ideals. If $\mathfrak{l}$ is a minimal left ideal, the author introduces the sum $\mathfrak{G}$ of all minimal left ideals which are isomorphic to $\mathfrak{l}$ (as $\mathfrak{o}$-modules) and proves that $\mathfrak{G}$ is a two sided ideal and is the direct sum of a two sided ideal whose square is 0 and a left ideal which is a simple ring. Doing the same for all types of nonisomorphic minimal left ideals and also for right ideals, a direct sum decomposition ("decomposition of the first degree") is obtained in which one term $\mathfrak{S}$ (the "residual part") is a two sided ideal and the other terms are simple subrings which are either left or right ideals. The "decomposition of the second degree" is obtained by taking the decomposition of the first degree of $\mathfrak{o}/\mathfrak{N}$, where $\mathfrak{N}$ is a certain nilpotent two sided ideal constructed with the help of nilpotent (left or right) ideals; the "decompositions of higher degree" are defined inductively.

C. Chevalley.

***Almeida Costa, A. Abelian groups, noncommutative rings and ideals, hypercomplex systems and representation. Vol. I.** Centro de Estudos de Mat. Fac. Ci. Pôrto. Publ. no. 3, 180 pp. (1942). (Portuguese) [MF 8782]

The first volume of this planigraphed book falls naturally in the main into two parts. The first part [chapters I–VI] is devoted to matrices and the solution of linear equations, the different chapters dealing with the situations arising in the cases of arbitrary fields, commutative fields, integral domains and rings with division algorithm. These chapters are followed by one devoted to the question of invariance of dimensionality treated in the author's pamphlet "On Abelian groups" [Anais Fac. Ci. Pôrto **27** (1942); these Rev. **4**, 267]. The second part, consisting of three chapters, deals with structure of rings in the style of the sixteenth chapter of the second edition of van der Waerden's "Moderne Algebra" [Springer, Berlin, 1930] but supplies more details and covers some literature not covered in that book as, for instance, Artin's theorem on primary rings.

G. Y. Rainich (Ann Arbor, Mich.).

Krull, Wolfgang. Beiträge zur Arithmetik kommutativer Integritätsbereiche. Eine Bemerkung zu den Beiträgen VI und VII. Math. Z. **48**, 530–531 (1942). [MF 8727]

The author corrects the proof of a theorem in his theory of ring extensions [see Math. Z. **45**, 319–334 (1939), in particular, pp. 321–322; these Rev. **1**, 37].

O. F. G. Schilling (Chicago, Ill.).

Krull, Wolfgang. Beiträge zur Arithmetik kommutativer Integritätsbereiche. VIII. Multiplikativ abgeschlossene Systeme von endlichen Idealen. Math. Z. **48**, 533–552 (1943). [MF 9005]

In an arbitrary integral domain $\mathfrak{R}$ let Λ be a multiplicatively closed system of finite ideals, that is, $\mathfrak{c}_1\varepsilon\Lambda$, $\mathfrak{c}_2\varepsilon\Lambda$ imply $\mathfrak{c}_1\cdot\mathfrak{c}_2\varepsilon\Lambda$ and each $\mathfrak{c}$ in Λ has a finite ideal basis. The author defines the Λ-operation on the ideals of $\mathfrak{R}$: $\mathfrak{a}\to\mathfrak{a}_\Lambda$, where $\mathfrak{a}_\Lambda$ is the sum (g.c.d.) of all ideals $\mathfrak{c}\varepsilon\mathfrak{R}$ for which there is an $\mathfrak{e}$ in Λ such that $\mathfrak{e}\cdot\mathfrak{c}\subset\mathfrak{a}$ ($\mathfrak{a}$ divides $\mathfrak{e}\cdot\mathfrak{c}$). The author's earlier definition [Beitrag I. p. 571, Math. Z. **41**, 545–577 (1936)] was based on a system of prime ideals instead of the present Λ. The new definition leads to simpler proofs and also to generalizations of known theorems to integral domains $\mathfrak{R}$ which are neither integrally closed nor satisfy the maximal condition (Teilerkettensatz). To a large extent the new results also apply to arbitrary commutative semigroups (Halbgruppen, ova) [cf. Lorenzen, Math. Z. **45**, 533–553 (1939); these Rev. **1**, 101]. The author discusses relations between the new Λ-operation and other ideal operations.

R. Hull (Lincoln, Neb.).

Krull, Wolfgang. Funktionaldeterminanten und Diskriminanten bei Polynomen in mehreren Unbestimmten. II. Monatsh. Math. Phys. **50**, 234–256 (1942). [MF 8544]

[The first part appeared in the same Monatsh. **48**, 353–368 (1939); these Rev. **1**, 102.] The present paper completes certain considerations from the first paper and adds various new results. The notations are as follows: K is a field which may be separable or inseparable; K^* is the algebraic closure of K. The polynomial rings in n variables over K are $P=K[x_1, \cdots, x_n]$ and $P^*=K^*[x_1, \cdots, x_n]$. The dimension d of a prime ideal p in P is the degree of transcendency of the residue ring P/p and $r=n-d$ is the dimension defect. The prime ideal p is said to be separable when by the decomposition of $p \cap P^*$ in primary components in P^* there is no proper primary component. It is shown that a prime ideal of defect r is separable only if one can determine r elements $a_1, \cdots, a_r$ in p such that for the functional determinant

$$\frac{\partial(a_1, \cdots, a_r)}{\partial(x_1, \cdots, x_r)}\not\equiv 0 \pmod{p}.$$

This result is a generalization of the fact that an irreducible polynomial in one variable is separable only when its derivative does not vanish identically.

Next the author considers systems of homogeneous forms in $n+1$ variables. So-called partial discriminants are introduced with the property that the partial discriminant for two variables vanishes for the multiple roots of the system of equations and also for those points for which the variables have the same value. The intersection of two different partial discriminants is the discriminant of the system. The partial discriminants for decomposable forms are studied. Of interest is the general result that for a field of characteristic two the discriminant of a system of forms is always the square of an irreducible form. There are various applications, some generalizations of the concept of a discriminant and calculations of discriminants for certain special types of forms.

O. Ore (New Haven, Conn.).

Schafer, R. D. Alternative algebras over an arbitrary field. Bull. Amer. Math. Soc. **49**, 549–555 (1943). [MF 8855]

A derivation is given of the result that the only central simple alternative and nonassociative algebra over an arbitrary field is the Cayley-Dickson algebra of order eight. Previous discussions had omitted the case where the characteristic is two or three. It is also shown that an isotope B with a unity quantity of an alternative algebra A is alternative, and that, if A is simple, B is isomorphic to A.

A. A. Albert (Chicago, Ill.).

NUMBER THEORY

Sispánov, Sergio. On the indetermined equation $ax^n+by^n+cz^n=0$. Bol. Mat. **16**, 71–73 (1943). (Spanish) [MF 9068]

Kanold, Hans-Joachim. Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl. J. Reine Angew. Math. **184**, 116–123 (1942). [MF 9039]

An odd perfect number must have the form

$$N=p^\alpha q_1^{2\beta_1}q_2^{2\beta_2}\cdots q_t^{2\beta_t}.$$

Steuerwald [S.-B. Bayer. Akad. Wiss. **1937**, 69–72] proved that the case $\beta_1=\beta_2=\cdots=\beta_t=1$ is impossible. The same was proved by the author for the cases $\beta_1=\beta_2=\cdots=\beta_t=2$ [J. Reine Angew. Math. **183**, 98–109 (1941); these Rev. **3**, 268] and for $\alpha=1$ or 5 and $\beta_1=\beta_2=\cdots=\beta_{t-1}=1$, $\beta_t=2$ [Deutsche Math. **4**, 53–57 (1939)]. Generalizing the last result, the author proves in this paper that the case $\beta_1=\beta_2=\cdots=\beta_{t-1}=1$, $\beta_t=2$ is always impossible. *A. Brauer.*

Kloosterman, H. D. Simultane Darstellung zweier ganzen Zahlen als einer Summe von ganzen Zahlen und deren Quadratsumme. Math. Ann. **118**, 319–364 (1942). [MF 8934]

The author studies the number $r_s(n, m)$ of solutions of the following system of Diophantine equations:

$$x_1^2+x_2^2+\cdots+x_s^2=n, \quad x_1+x_2+\cdots+x_s=m,$$

where n and m are positive integers. The Hardy-Littlewood methods easily apply to give approximations for $r_s(n, m)$ even when the sum of squares is replaced by any positive definite form with integral coefficients and the linear form by any linear form with integral coefficients; but the author's aim is to use these approximations to get exact results for certain special cases. Using methods based on those of Mordell and Hardy [L. J. Mordell, Quart. J. Math. **48**, 93–104 (1917); G. H. Hardy, Trans. Amer. Math. Soc. **21**, 255–284 (1920)] he proves certain results leading to the following three theorems. For $s=3, 5, 7$ it is true that $r_s(n, m)\neq 0$ only if $m^2\leqq sn$ and $m\equiv n \pmod 2$. In case

$m^2=sn$, then $r_s(n, m)=1$. In case $m^2<sn$ and $m\equiv n \pmod 2$, then

$$r_3(n, m)=3\sum_{d|N}\left(\frac{d}{3}\right),$$

$$r_5(n, m)=5N\sum_{d|N}\left(\frac{d}{5}\right)\cdot\frac{1}{d},$$

$$r_7(n, m)=\frac{7N^2}{8}\sum_{d|N}\left(\frac{d}{7}\right)\cdot\frac{1}{d^2}$$

for $N\not\equiv 0 \pmod s$ and

$$r_3(n, m)=6\sum_{d|N}\left(\frac{d}{3}\right),$$

$$r_5(n, m)=5N\left(1-\left(\frac{N_0}{5}\right)\cdot\frac{1}{5^\beta}\right)\sum_{d|N_0}\left(\frac{d}{5}\right)\cdot\frac{1}{d},$$

$$r_7(n, m)=\frac{7N^2}{8}\left(1-\left(\frac{N_0}{7}\right)\cdot\frac{1}{7^{2\beta}}\right)\sum_{d|N_0}\left(\frac{d}{7}\right)\cdot\frac{1}{d^2}$$

for $N\equiv 0 \pmod s$, where β is the highest power of s dividing $N=\frac{1}{2}(sn-m^2)$ and $N=s^\beta N_0$, while the sums range over all positive divisors of N and N_0, respectively. By letting m vary, new formulas for the sum of 3, 5 and 7 squares are obtained. *B. W. Jones* (Ithaca, N. Y.).

Kesava Menon, P. Some theorems on residues. Proc. Indian Acad. Sci., Sect. A. **17**, 107–113 (1943). [MF 8786]

Referring to a statement of Hardy and Wright in their book "Introduction to the Theory of Numbers" [Oxford University Press, Oxford, 1938], the author intends to improve the results on the residues of $(a^{p-1}-1)/p \pmod p$. Hardy and Wright [l.c., pp. 104–105] prove only the theorem of Eisenstein on $(2^{p-1}-1)/p$, but in a footnote [p. 105] they say that references to later proofs and generalizations will be found in Dickson's "History of the Theory of Numbers" [vol. 1, Stechert, New York, 1934, ch. 4]. The author does not mention any of these papers. He proves the following theorem. Let m, n and d be integers such that $m-1=nd$. Then

$$\frac{d^{\varphi(m)}-1}{m}\equiv\frac{1}{d}\sum\left(\left\{\frac{x}{n}\right\}\Big/x\right) \pmod m,$$

where $\{x\}$ denotes the smallest integer greater than or equal to x, and where x runs over the integers less than m and relatively prime to m. He obtains a similar result for the case $m+1=nd$. In this form these results cannot be found explicitly in any earlier paper, but it can easily be proved that they are really special cases of a theorem stated by Sylvester [Collected Mathematical Papers, vol. 2, Cambridge University Press, Cambridge, England, 1908, pp. 229–235, 241, 262–263] for the case where m is a prime, corrected by Mirimanoff [J. Reine Angew. Math. **115**, 295–300 (1895)] and generalized by H. F. Baker [Proc. London Math. Soc. (2) **4**, 131–135 (1906)] for an arbitrary m as follows. If m' is determined by $mm'\equiv 1 \pmod a$ and if $(m'x)$ denotes the least positive residue of $m'x \pmod a$, then

$$\frac{a^{\varphi(m)}-1}{m}\equiv\sum\frac{(m'x)}{m-x} \pmod m,$$

where x runs over the positive integers less than m and relatively prime to m. Moreover, the author proves for odd integers m that

$$\prod a\equiv(-1)^{\varphi(m)/2}\left(\prod b\right)^2\cdot 2^{2\varphi(m)} \pmod{m^2},$$

where a runs over the positive integers less than m and relatively prime to m and where b runs over those of these integers which are less than $m/2$. A similar result is proved for $m\not\equiv 0 \pmod 3$. The remaining results of the paper are without any interest since the residue of $d^{\varphi(m)/2} \pmod m$ can easily be found by the use of Jacobi's symbols. *A. Brauer* (Chapel Hill, N. C.).

Kesava Menon, P. Multiplicative functions which are functions of the g.c.d. and l.c.m. of the arguments. J. Indian Math. Soc. (N.S.) **6**, 137–142 (1942). [MF 8297]

An arithmetic function $f(M_1, M_2, \cdots, M_r)$ is called multiplicative if

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r)=f(M_1, M_2, \cdots, M_r)f(N_1, N_2, \cdots, N_r)$$

for $(M_1M_2\cdots M_r, N_1N_2\cdots N_r)=1$. The author generalizes two theorems of Vaidyanathaswamy [Trans. Amer. Math. Soc. **33**, 579–662 (1931)] on multiplicative functions. *A. Brauer* (Chapel Hill, N. C.).

Kesava Menon, P. Transformations of arithmetic functions. J. Indian Math. Soc. (N.S.) **6**, 143–152 (1942). [MF 8298]

An arithmetic multiplicative function is called linear by Vaidyanathaswamy and by the author if the equation $f(MN)=f(M)f(N)$ holds always. Suppose that

$$M=p_1^{m_1}p_2^{m_2}\cdots p_r^{m_r},\quad N=p_1^{n_1}p_2^{n_2}\cdots p_r^{n_r}.$$

Set

$$T(M, N)=\prod_{i=1}^{r}\binom{m_i-1}{n_i-1},\quad T(1, 1)=1.$$

Then $T(M, N)=0$ [instead of $\neq 0$ as in the paper] unless N is a divisor of M and is divisible by each of the primes $p_1, p_2, \cdots, p_r$. Let $L_1(M)$ and $L_2(M)$ be linear arithmetic functions and $\lambda_k(M)=k^t$, where t is the number of all the prime factors of M. If $f(M)$ is a multiplicative function, and if

$$f_1(M)=\sum_{d|M}L_1(M/d)L_2(d)T(M, d)f(d),$$

then it is proved in this paper that

$$f(M)=(L_2(M))^{-1}\sum_{d=1}^{M}L_1(M/d)\lambda_{-1}(M/d)T(M, d)f_1(d).$$

Moreover, the author proves some similar results. *A. Brauer* (Chapel Hill, N. C.).

Lehner, Joseph. Ramanujan identities involving the partition function for the moduli 11^α. Amer. J. Math. **65**, 492–520 (1943). [MF 8706]

Rademacher [Trans. Amer. Math. Soc. **51**, 609–636 (1942); these Rev. **3**, 271] has developed a systematic method for the investigation of Ramanujan identities and applied it to the moduli 5^α, 7, 7^2, 13^α. In the present paper the author establishes identities for the moduli 11^α in a form involving only modular functions. When expressed in terms of the partition function the identity for the modulus 11, for example, takes the form

$$\sum_{n=0}^{\infty}p(11n+6)x^{n+1}=11\prod_{m=1}^{\infty}(1-x^{11m})^{-1}\{11AC^2-11^2C+2AC-32C-2\}.$$

It follows that the Ramanujan conjecture $p(n)\equiv 0 \pmod{q^\alpha}$ if $24n\equiv 1 \pmod{q^\alpha}$ is true for $q=11$, $\alpha=1$.

The general result for powers of 11 is the following theorem: for every integer $\alpha\geqq 1$ there is a "Ramanujan identity" of the form

$$\text{(A)}\qquad L(\tau; 11^\alpha)=P(A, C),$$

where L is defined in terms of the linear operator

$$U_q F(\tau) = q^{-1} \sum_{\lambda=0}^{q-1} F((\tau+\lambda)q^{-1})$$

by means of the equations

$$L(\tau; q^{2\beta+1}) = U_q\{\Phi(\tau; q^2)L(\tau; q^{2\beta})\},$$
$$L(\tau; q^{2\beta+2}) = U_q L(\tau; q^{2\beta+1}),$$
$$\Phi(\tau; q^2) = \eta(q^2\tau)/\eta(\tau),$$
$$\eta(\tau) = e(\tau/24) \prod_{m=1}^{\infty} (1-x^m), \qquad x = e(i\tau);$$

$A = A(\tau)$, $C = C(\tau)$ are certain entire modular functions having power series expansions in $x = e(i\tau)$ with integral coefficients; and P is a polynomial in A and C with rational coefficients.

The author was not able to prove that the coefficients of P were integral except for $\alpha = 1, 2$. Thus for $\alpha > 2$ the identity (A) does not, as yet, yield any information concerning the Ramanujan conjecture for $q = 11^\alpha$.
R. D. James (Vancouver, B. C.).

Rademacher, Hans. On the expansion of the partition function in a series. Ann. of Math. (2) **44**, 416–422 (1943). [MF 8872]

In the usual treatment of the partition function [G. H. Hardy and S. Ramanujan, Proc. London Math. Soc. (2) **17**, 75–115 (1918); H. Rademacher, Proc. London Math. Soc. (2) **43**, 241–254 (1937)] the path of integration used is the circle $|x| = \exp(-2\pi N^{-2})$, which is then broken up by the Farey dissection of order N. Here the author replaces the path by that given by $x = \exp(2\pi i\tau)$, where τ goes from i to $i+1$ along the upper arcs of a series of circles with centers at $h/k + i/2k^2$ and radii $1/2k^2$, where h/k runs through the Farey series of order N. That these circles give a suitable connected path follows from a theorem of L. R. Ford [Amer. Math. Monthly **45**, 586–601 (1938)]. The advantage of this new path of integration is that it simplifies considerably the estimates that must be made and it clarifies the manner in which the singularities at the various rational points contribute to the final formula. The author carries out the proof for the evaluation of the partition function but his method will be advantageous in similar problems involving modular functions of nonnegative dimension. *H. S. Zuckerman* (Seattle, Wash.).

Lehmer, D. H. Ramanujan's function $\tau(n)$. Duke Math. J. **10**, 483–492 (1943). [MF 8970]

In trying to disprove Ramanujan's hypothesis that $|\tau(p)| < 2p^{11/2}$, p a prime, the author computed the values of $\tau(p)$ for $p < 300$ and $p = 571$ without finding a case in which the hypothesis is false. The reason for testing $p = 571$ is not explained. The values of $\tau(n)$ for all $n \leqq 300$ are tabulated and they are discussed in relation to questions arising from the problem of the order of $\tau(n)$. In this connection the numerical value of Rankin's formula for $\lim_{n\to\infty} \sum_{\nu \leqq n} \tau(\nu)^2$ is determined. *H. S. Zuckerman.*

Peng, H. Y. A result in divisor problem. Acad. Sinica Science Record **1**, 69–72 (1942). [MF 8839]

It was proved by Walfisz [Ann. Scuola Norm. Super. Pisa (2) **5**, 289–298 (1936)] that, if

$$\tau(x) = \sum_{n \leqq x} (x-n) \sum_{d|n} 1/d,$$

then

$$\tau(x) = (\pi^2/12)x^2 - \tfrac{1}{2}x \log x - \tfrac{1}{2}(\gamma - 1 + \log 2\pi)x + R(x),$$

where $R(x) = O(x^{3/10})$. Here van der Corput's method is used to prove that $R(x) = O(x^{2/7} \log x)$. *E. C. Titchmarsh.*

Buchstab, A. A. On a relation for the function $\pi(x)$ expressing the number of primes that do not exceed x. Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 152–160 (1943). (Russian. English summary) [MF 8800]

The author derives in an elementary way an "inversion" of Legendre's celebrated formula [misquoted in the paper] for the number $\pi(x)$ of primes not exceeding x, namely,

$$\pi(x) - \pi(x^{\frac{1}{2}}) + 1 = \sum [x/k]\mu(k),$$

where $\mu(k)$ is Möbius' function and the sum proceeds over all k's having no prime factor exceeding $x^{\frac{1}{2}}$. The author's inverted formula is

$$x - 1 = \sum_{j=1}^{\infty} (-1)^{j-1} \sum_k \binom{\pi((x/k)^{1/j})}{j}$$

where the inner sum of binomial coefficients extends over all k's having no prime factor exceeding $(x/k)^{1/j}$. The author goes on to show that (unfortunately) there may be nearly as many nonzero terms in this sum as primes not exceeding x. *D. H. Lehmer* (Berkeley, Calif.).

Vijayaraghavan, T. On the fractional parts of the powers of a number. III. J. London Math. Soc. **17**, 137–138 (1942). [MF 8275]

Let $\theta > 1$ and assume that the fractional part of θ^n $(n = 1, 2, \cdots)$ has only a finite number of limit points. It is conjectured that any such θ is algebraic [Proc. Cambridge Philos. Soc. **37**, 349–357 (1941); these Rev. **3**, 274] and it is proved that their number is denumerable. The limit points can be enclosed in one of a denumerable sequence of sets T of intervals of maximum length δ; let Δ be the minimum interval of the complement of T in $(0, 1)$. It can be assumed that $\Delta/\delta > 4m+4$, where $m = [\theta]$, and that θ^n belongs to T if $n \geqq q$. If now θ' has the same property, $[\theta'] = m$ and $0 < \theta - \theta' \leqq \delta q^{-1}(m+1)^{1-q}$, then $\theta^q - \theta'^q < \delta$. Let N be the least positive integer for which $\theta^{N+1} - \theta'^{N+1} > 2\delta$, whence $N \geqq q$; then $\theta^{N+1} - \theta'^{N+1} > \Delta > (4m+4)\delta > (\theta + \theta') \cdot 2\delta \geqq (\theta+\theta')(\theta^N - \theta'^N) > \theta^{N+1} - \theta'^{N+1}$, a contradiction. Hence $\theta - \theta' > \delta_q^{-1}(m+1)^{1-q}$; and the number of all θ's is denumerable.
G. Pall (Montreal, Que.).

ANALYSIS

Theory of Functions of Complex Variables

Vigil, Luis. On Taylor series with, or without, analytic continuation. The present state of Borel's theorem. Revista Mat. Hisp.-Amer. (4) **3**, 137–144 (1943). (Spanish) [MF 8952]

The paper is to be continued. This first installment reviews known results.

Beckenbach, E. F. The stronger form of Cauchy's integral theorem. Bull. Amer. Math. Soc. **49**, 615–618 (1943). [MF 8889]

The author gives a relatively short and simple proof of the stronger form of Cauchy's integral theorem: if $f(z)$ is holomorphic on the interior D of a simply closed rectifiable curve C, and continuous on $D+C$, then $\int_C f(z)dz = 0$. The proof uses the Osgood-Carathéodory form of Riemann's

mapping theorem. The method is applicable to the establishment of the stronger form of Green's lemma, etc.

W. T. Martin (Syracuse, N. Y.).

Rogosinski, Werner. On the coefficients of subordinate functions. Proc. London Math. Soc. (2) **48**, 48–82 (1943). [MF 9112]

Let D be a simply-connected region of the extended z-plane and let $w=F(z)$ be meromorphic in D. Let $w=f(z)$ denote a second function meromorphic in D. Then $f(z)$ is called subordinate to $F(z)$ in D with center $z_0(\epsilon D)$ if $f(z_0)=F(z_0)$ and if the values w of $f(z)$ for z in D determined by analytic continuation from z_0 are situated in the Riemannian image of D with respect to $F(z)$; if further the Riemannian image of D with respect to F is a continuation of the Riemannian image of D with respect to $f(z)$, then $f(z)$ is called schlicht subordinate to $F(z)$ in D with center z_0. Under the normalizing assumptions that z_0 is 0 and $f(0)=F(0)=0$, $F(z)$ and $f(z)$ are written as $\sum_{k=1}^{\infty}A_kz^k$ and $\sum_{k=1}^{\infty}a_kz^k$, respectively, these expansions being valid in the neighborhood of $z=0$. The paper is concerned with finding relations between the a_k and A_k implied by the subordination of $f(z)$ to $F(z)$ in D with center at 0. Two principal problems are distinguished: (I) given $F(z)$, to determine an estimate for $|a_k|$ for fixed k and the whole class of subordinated $f(z)$; (II) given $F(z)$ and a fixed function $f(z)$ subordinate to $F(z)$, to determine estimates for the sequence $\{|a_k|\}$.

Among the results obtained the following may be mentioned. The region D is assumed to be the interior of the unit circle. Then $|a_n|\leqq n^{\frac{1}{2}}\max_{1\leqq k\leqq n}(|A_k|)$ and the order $n^{\frac{1}{2}}$ is in general best possible for problems (I) and (II) even in the case of schlicht subordination. Further, it is shown that

$$(1)\qquad \sum_{k=1}^{m}|a_k|^2\leqq\sum_{k=1}^{m}|A_k|^2,\qquad m=1, 2, \cdots,$$

and (2) in the case of schlicht subordination that

$$\sum_{k=1}^{m}k|a_k|^2\leqq\sum_{k=1}^{m}k|A_k|^2,\qquad m=1, 2, \cdots.$$

If $A_1, A_2, \cdots, A_n$ are nonnegative, nonincreasing and convex, then $|a_n|\leqq A_1$. If $A_1, A_2, \cdots, A_n$ are nonnegative, nondecreasing and convex, then $|a_n|\leqq A_n$. If $F(z)$ is schlicht and either the A_k are real or the image of $|z|<1$ under $F(z)$ is starlike with respect to 0, then $|a_n|\leqq n|A_1|$ $(n=1, 2, \cdots)$. Among other results, too numerous to mention, is the following: there exist a function $F(z)$ with $A_k=0(1)$ and $f(z)$ schlicht subordinate to $F(z)$ such that a_k is not $o(k^{\frac{1}{2}})$.

The proofs are based upon a variety of methods such as that of Littlewood [Proc. London Math. Soc. (2) **23**, 481–519 (1924)], I. Schur [J. Reine Angew. Math. **147**, 205–232 (1917); **148**, 122–145 (1918)] and the anterior work of the author which lies in the direction of the present paper.

M. H. Heins (Chicago, Ill.).

Zuckerman, Herbert S. Certain functions with singularities on the unit circle. Duke Math. J. **10**, 381–395 (1943). [MF 8475]

In the method of determining the Fourier coefficients of modular functions of positive dimension [Rademacher and Zuckerman, Ann. of Math. (2) **39**, 433–462 (1938)] the group-theoretical property of the modular functions is used only for obtaining their asymptotic behavior on the Farey arcs. If we prescribe this behavior directly we can still formally carry out the process for obtaining coefficients of an expansion. Whether, however, there exists at all any function of the desired behavior remains uncertain, and thus it is a problem whether the formally obtained expansion shows the required asymptotic properties.

The author shows first through an example that a certain limitation has to be set to the preassignment of properties of $\sum_{n=-\mu}^{\infty}a_nx^n$ on the Farey arcs. Within these limitations, however, he proves four theorems, the type of which may be illustrated by the following specialization of his second theorem. Let r be a nonnegative integer, $\alpha\geqq0$, $\beta\geqq2$, $B\geqq0$, and let $d_{h,k}$ be a set of real or complex numbers defined for $0\leqq h<k$, $(h, k)=1$, such that $d_{h,k}=O(k^{r\beta-2-\epsilon})$, $\epsilon>0$. Put

$$f(x)=\sum_{m=0}^{\infty}a_mx^m$$

with

$$a_m=2\pi i\sum_{k=1}^{\infty}A_k(m)(B/(k^\beta(\alpha+m)))^{(r+1)/2}I_{r+1}(2(Bk^{-\beta}(\alpha+m))^{\frac{1}{2}}),$$

$$A_k(m)=\sum_{\substack{h \bmod k\\(h,k)=1}}d_{h,k}e^{-2\pi imh/k};$$

then we have

$$f(\exp(-2\pi N^{-2}+2\pi i(\varphi+p/q))=d_{p,q}i(2\pi)^{r+1}(N^{-2}-i\varphi)^r$$
$$\times\exp(B(2\pi)^{-1}q^{-\beta}(N^{-2}-i\varphi)^{-1}+2\pi\alpha(N^{-2}-i\varphi))+O(1),$$

as $N\to\infty$, for $\varphi+p/q$ on the Farey interval of order N about p/q. That in general the term $O(1)$ cannot be replaced by $o(1)$ is shown by the above mentioned example.

H. Rademacher (Swarthmore, Pa.).

Fuchs, W. H. J. A uniqueness theorem for mean values of analytic functions. Proc. London Math. Soc. (2) **48**, 35–47 (1943). [MF 9111]

Let

$$M_p(r, f)=\left\{(1/2\pi)\int_{-\pi}^{\pi}|f(re^{i\theta})|^pd\theta\right\}^{1/p}.$$

The author establishes the following conjecture of A. E. Ingham. Let $f(z)$ and $g(z)$ be single valued and analytic in $a<|z|<b$, and let $M_p(r, f)=M_p(r, g)$ for some p, $1<p<\infty$, and for an infinite number of values of r having a limit point R, $a<R<b$. Then $M_p(r, f)=M_p(r, g)$ for $a<r<b$. (Simple examples show that the result fails when $p=1$ or $p=\infty$.) If f and g have no zeros in $a<|z|<b$, the theorem is easily proved by considering $f^{p/2}$ and $g^{p/2}$; then $M_2^2(r, f^{p/2})$ and $M_2^2(r, g^{p/2})$ are analytic functions of the complex variable r, and the result follows. In the general case, $f^{p/2}$ and $g^{p/2}$ are analytic only in each ring of a set $a_n<|z|<b_n$, and it is necessary to study the function in detail as a function of the complex variable r and show that its values in one ring determine its values in the next.

R. P. Boas, Jr. (Cambridge, Mass.).

Wolff, Julius. Sur les fonctions holomorphes univalentes. C. R. Acad. Sci. Paris **213**, 158–160 (1941). [MF 9159]

The hypotheses of Fatou's classical theorem on functions analytic and bounded in the interior of the unit circle are specialized by requiring that the mapping be schlicht as well. The following theorem is obtained. Let Γ_α denote a rectifiable Jordan arc lying in $|z|<1$ except for its endpoint α on $|z|=1$ and having the property that all its chords make angles with the radius joining the origin to α of magnitude less than $\pi/2-\epsilon$ $(0<\epsilon<\pi/2$, ϵ otherwise arbitrary but fixed). If $f(z)$ is analytic, bounded and schlicht for $|z|<1$, then for almost all points α of the circumference of

the unit circle, the image of Γ_n with respect to $f(z)$ is rectifiable. *M. H. Heins* (Chicago, Ill.).

Loomis, Lynn H. On an inequality of Seidel and Walsh. Bull. Amer. Math. Soc. 48, 908–911 (1942). [MF 7510]

In a recent paper [Trans. Amer. Math. Soc. 52, 128–216 (1942); these Rev. 4, 215] Seidel and Walsh introduced the concept of the radius of p-valence of a Riemann surface at a point w_0. Let C_p denote a p-sheeted circle of center w_0 and radius ρ of a Riemann surface R, that is, a simply connected region of R which lies over the circle $|w-w_0|<\rho$ and covers it precisely p times. Given a point w_0 of R, let ρ_p be the radius of the largest C_p with center w_0; if none exists, let $\rho_p=0$. The radius $D_p(w_0)$ of p-valence of R at w_0 is then defined as the maximum of the ρ_n for $n\leqq p$.

Let $w=f(z)=a_1z+a_2z^2+\cdots$ be analytic and $|f(z)|<M$ for $|z|<1$ and let R denote the Riemann surface on to which $f(z)$ maps the circle $|z|<1$. Let w_0 be the image of $z=0$; w_0 lies over $w=0$. Seidel and Walsh proved among other results that there exist two constants λ_p, depending only on p, and Λ_p, depending on p and M, such that

$$\lambda_p D_p(w_0)\leqq\sum_{n=1}^{p}|a_n|\leqq\Lambda_p[D_p(w_0)]^{2^{-p}},$$

where $\Lambda_p=24_pM^r$, $r=1-2^{-p}$. The author of the present paper shows: (1) the exponent 2^{-p} may be replaced by $1/(p+1)$, and this is the best possible exponent (for $D_p\to0$); (2) λ_p may be replaced by 1, and this value of the constant is the best possible. *S. E. Warschawski.*

Ferrand, Jacqueline. Sur la représentation conforme au voisinage d'un point frontière. C. R. Acad. Sci. Paris 212, 977–980 (1941). [MF 9179]

Let Δ denote a simple-connected region of the ζ-plane admitting an accessible frontier point π_0 at infinity belonging to a prime end E, of which it is the sole accessible point. The region Δ is mapped (1, 1) and conformally onto the right-half z-plane by $\zeta=\varphi(z)$ in such a manner that $z=\infty$ corresponds to the prime end E. It is shown that, if Δ contains an angular sector S_0 ($|\zeta|>R_0$, $\theta_1<\arg\zeta<\theta_2$), then $\zeta=\varphi(z)$ tends to the unique limit $\zeta=\infty$ when z tends to infinity in $|\arg z|<\pi/2-\epsilon$ ($0<\epsilon<\pi/2$, ϵ otherwise arbitrary and fixed). Related theorems, whose statements are of a more complicated character, are also given. Use is made of the theory of harmonic measure and the work of Ostrowski [Acta Math. 64, 81–184 (1935).] *M. H. Heins* (Chicago, Ill.).

Rengel, Ewald. Verzerrung des Randes bei schlichter konformer Abbildung. I. Deutsche Math. 6, 370–378 (1942). [MF 8612]

Rengel, Ewald. Verzerrung des Randes bei schlichter konformer Abbildung. II. Deutsche Math. 6, 379–393 (1942). [MF 8613]

Using elementary mappings of the type $w=z+1/z$, the author derives a number of results from the theorem of Bieberbach [S.-B. Berlin. Math. Ges. 38, 940–955 (1916)], which the author incorrectly ascribes to H. Grötzsch, that the boundary of the image R of $|z|>1$ under a schlicht map $w=z+a_1z^{-1}+\cdots+a_nz^{-n}+\cdots$ lies in $|w|\leqq2$. In the first paper precise limits are obtained for the diameter of the boundary of R on the real axis under the assumption that the complement of R contains a special type of region symmetric in the real axis, such as a circle or an ellipse. In the second paper conditions are studied under which the complement of R contains a straight line segment symmetric in the real axis. The most easily stated of the author's results is the following theorem. Let $w=u+iv=f(z)$ give a normed schlicht conformal map of $|z|<1$. Then on each line $u=c$ ($-\frac{1}{2}\leqq c\leqq\frac{1}{2}$) at least one image point lies outside the circumference $|w|=\frac{1}{2}$ or else $f(z)$ is the function which maps $|z|<1$ onto the plane slit along the part of $u=c$ outside of $|w|<\frac{1}{2}$. *L. H. Loomis* (Cambridge, Mass.).

Weyl, Hermann. On Hodge's theory of harmonic integrals. Ann. of Math. (2) 44, 1–6 (1943). [MF 8070]

In chapter III of his book [The Theory and Applications of Harmonic Integrals, Cambridge University Press, Cambridge, England, 1941; these Rev. 2, 296] W. V. D. Hodge attempts to establish the existence of harmonic integrals with preassigned periods. His proof is not conclusive since it is based partially on a false statement [p. 136] concerning the behavior of the solution of a nonhomogeneous integral equation when the parameter approaches a proper value. In the present paper a correct proof is given. It is based on the same formal foundations laid down by Hodge, but it requires a more careful consideration of the solutions of the homogeneous integral equation connected with the problem. *F. Bohnenblust* (Princeton, N. J.).

Teichmüller, Oswald. Drei Vermutungen über algebraische Funktionenkörper. J. Reine Angew. Math. 185, 1–11 (1943). [MF 8813]

In a field of algebraic functions of one variable one can consider the additive group of differentials ωdz^m of a given dimension m, m a rational integer. In this group the differentials which are finite at a given place $\mathfrak{p}$ of the field form a subgroup. The corresponding difference classes are termed by the author "principal parts" (Hauptteil) for the differentials of dimension m at the place $\mathfrak{p}$. He speaks of a system of principal parts (abbreviation: s.p.p.) if a principal part is given for each place $\mathfrak{p}$, provided that only for a finite number of places the principal part is different from zero. In the additive group of all s.p.p. the s.p.p. which are defined by differentials of dimension m form a subgroup, and the corresponding difference classes are termed classes of s.p.p.; these form a module over the field of constants.

Let $d\zeta_1^{1-m}, d\zeta_2^{1-m}, \cdots, d\zeta_\tau^{1-m}$ be a basis for the differentials of dimension $1-m$ of the first kind (that is, which are everywhere finite; the symbol $d\xi^m$ does not indicate an mth power). If $\omega^{(m)}$ denotes a s.p.p. for the dimension m, the expression

$$f_i=\sum_{\mathfrak{p}}\operatorname{Res}_{\mathfrak{p}}(\omega^{(m)}d\zeta_i^{1-m})$$

has an obvious meaning. If $\omega^{(m)}=d\xi^m$, then $f_i=0$. Hence f_i is a linear function on the module of classes of s.p.p. The author proves that these τ functions f_i are linearly independent and that they all vanish simultaneously only at the zero element of the module of classes of s.p.p. Hence there is a duality (or a correlation) between the module of classes of s.p.p. for dimension m and the module of differentials of the first kind of dimension $1-m$.

The euristic source of the conjectures stated below is the consideration of a field Z_y of algebraic functions of a complex variable z, where the field Z_y depends algebraically (or analytically) on a complex parameter y. As y varies, Z_y traces out an analytical curve on the manifold R of classes of birationally equivalent curves of a given genus g, unless Z_y remains constantly isomorphic to a fixed function field, in which case the image of the variable field Z_y is a point

of R. The author has shown elsewhere [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 22; these Rev. **2**, 187] that there exists a duality between the module of tangential vectors of R at a point Z_y and the module of quadratic differentials of the first kind in Z_y. Hence there must also be an isomorphism between the former and the module of classes of s.p.p. in Z_y for the dimension -1. The author calculates explicitly the class of s.p.p. which corresponds in the above isomorphism to the particular tangential vector at Z_y defined by the given one-dimensional algebraic family of fields Z_y. If Z_y remains constantly isomorphic to a fixed field, that vector must be the zero vector, and the corresponding class of s.p.p. must be the zero class. On the other hand, it stands to reason that the isomorphism, or the birational transformation, between Z_y and the fixed field depends algebraically on y. On the basis of these euristic considerations the author is led to the following two conjectures.

Let k be an algebraically closed field of characteristic zero, Y a field of algebraic functions over k, K a field of algebraic functions having Y exactly as field of constants (that is, Y is algebraically closed in K). Let y be an element of Y, not in k, and let z be an element of K, not in Y. For any place $\mathfrak{P}$ of K/Y let z' be a local uniformizing coordinate (that is, z' is an element of K which is finite at $\mathfrak{P}$ and has ramification order 1), and let $F(z, z')=0$, where F is an irreducible polynomial with coefficients in Y. The differential expression $-F_y/F_z\cdot dz^{-1}$ defines a s.p.p. in K/Y which is independent of z' and of F. First conjecture: the class of s.p.p. defined by $-F_y/F_z\cdot dz^{-1}$ is independent of z. Second conjecture: the condition that the above class be the zero class is necessary and sufficient in order that the following be true: there exists a finite algebraic extension Y^* of Y such that the corresponding extension K^* of K ($K^*=Y^*K$) be generated by Y^* and by a field X of degree of transcendency 1 over k. *O. Zariski* (Baltimore, Md.).

Eichler, M. Bemerkungen zu den vorstehenden Vermutungen von Teichmüller. J. Reine Angew. Math. **185**, 12–13 (1943). [MF 8814]

The first of the two conjectures formulated by Teichmüller [see preceding review] is proved in a quite elementary fashion. The second conjecture is proved for hyperelliptic fields (that is, K/Y is hyperelliptic).

O. Zariski (Baltimore, Md.).

Functional Analysis, Ergodic Theory

Šmulian, V. Sur les ensembles compacts et faiblement compacts dans l'espace du type (B). Rec. Math. [Mat. Sbornik] N.S. **12(54)**, 91–98 (1943). (French. Russian summary) [MF 8795]

Let E be a Banach space, $\bar{E}$ its conjugate space. A set $M\subset\bar{E}(S\subset E)$ is compact with respect to $S\subset E$ ($M\subset\bar{E}$) if from every sequence $\{f_n\}\subset M$ ($\{x_n\}\subset S$) it is possible to select a subsequence $\{f_{n_i}\}$ ($\{x_{n_i}\}$) such that $\lim f_{n_i}(x)$ ($\lim f(x_{n_i})$) exists uniformly for $x\epsilon S$ ($f\epsilon M$). In a second definition one replaces "compact" by "weakly compact" and "uniformly" by "quasi-uniformly." The principal theorem is the following. Let $S\subset E$, $M\subset\bar{E}$ be bounded sets. Then the relation of one set being strongly or weakly compact with respect to the other is reciprocal. Applications are made to obtain known theorems of Banach, Gelfand, Phillips, Sirvint and others. *J. V. Wehausen.*

Šmulian, V. On some problems of the functional analysis. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 157–159 (1943). [MF 8681]

Let E be a Banach space and $\bar{E}$ the space conjugate to E. The author provides counterexamples to show that the following two hypotheses are not generally valid. (A) If a bounded set $M\subset\bar{E}$ is closed in the weak topology of $\bar{E}$ defined by elements of E and if $F\epsilon\bar{E}$ is continuous in this topology on M, then for any $\epsilon>0$ there exists $x_\epsilon\epsilon E$ such that $|F(f)-f(x_\epsilon)|\leqq\epsilon$ for $f\epsilon M$. (B) Let Q be a bicompact Hausdorff space. Suppose that a partially ordered sequence of continuous functions $\{\varphi_\alpha(q)\}$, $\|\varphi_\alpha\|\leqq1$, converges to zero at every point of Q. Then $f(\varphi_\alpha)\to0$ for any linear functional f on the space of continuous functions on Q. Let $\psi_\alpha(t)$ be a partially ordered sequence of bounded functions on an abstract set T. Theorem: if $\psi_\alpha\to0$ weakly, then

$$\lim_\alpha\lim_\beta|\psi_\alpha(t_\beta)|=0,$$

where $\{t_\beta\}$ is a partially ordered sequence from T. Conversely, if B is valid for the system α, then every uniformly bounded partially ordered sequence satisfying the equation above converges weakly to zero. *J. V. Wehausen.*

Mackey, George W. On infinite dimensional linear spaces. Proc. Nat. Acad. Sci. U. S. A. **29**, 216–221 (1943). [MF 8716]

Let X denote a real linear space without a topology, an let X^* denote the linear space of all real linear (distributive) functionals defined on X. This paper sketches, without proofs, a method of introducing structure into X. It is observed that, with certain "uninteresting" exceptions, corresponding to each subspace L of X^* a topology may be introduced so as to make X a linear topological space with L as exactly the class of continuous linear functionals defined on X. The space X, together with any given subspace L of X^*, is said to form a linear system X_L. The notion of a closed subspace of X_L is defined, and the structure of X_L is investigated in terms of the lattice of all its closed subspaces. The notion of a bounded set in X_L is defined, and, in terms of this, the phrases "L is boundedly closed," "L is relatively bounded" are given meaning. Completeness of a linear system is defined in such a way that, if L is a total set of functionals, X_L is complete if and only if every total subspace of L contains L in its bounded closure. With these notions it is possible to study the question of when L is a norm set, that is, of when X is a normable linear space with L as the space of continuous linear functionals defined on it. One theorem is that, if L is total and X_L is complete, then L is a norm set if and only if L is relatively bounded and boundedly closed. Numerous other notions and applications are indicated. *A. E. Taylor* (Los Angeles, Calif.).

Aronszajn, N. La théorie des noyaux reproduisants et ses applications. I. Proc. Cambridge Philos. Soc. **39**, 133–153 (1943). [MF 9196]

Here F is a class of real or complex valued functions f on a general set E, which satisfies the conditions of a general Hilbert space (no restrictions on dimension). The objective is to study reproducing kernels $N(x, y)$, that is, such that $f(y)=[f(x), N(x, y)]_x$ for all y of E and f of F, where $[f, g]$ is the fundamental scalar product on F. If such a kernel N exists then (a) it is a semidefinite positive Hermitian function, that is,

$$N(x, y)=\overline{N(y, x)},\quad \sum_{i=1}^n\sum_{j=1}^n N(x_i, x_j)\xi_i\bar{\xi}_j\geqq0$$

for all sets $x_1, \cdots, x_n$ of E and numbers $\xi_1, \cdots, \xi_n$; (b) the functions $u_y(x) = N(x, y)$ belong to F for each y and form a complete set of functions on F. A necessary and sufficient condition that a reproducing kernel on F exist is that $f(y)$ considered as a function on F, for each y, be a linear functional on F. A given kernel $N(x, y)$ on the product set EE gives rise to a general Hilbert space F for which N is a reproducing kernel if and only if it is a semidefinite positive Hermitian function. It might be mentioned that the sufficiency of this condition is to be found in Moore's "General Analysis" [Mem. Amer. Philos. Soc., vol. I, pt. II, pp. 79 and 101]. Examples of spaces with reproducing kernels mentioned include sequences of convergent square, absolutely continuous functions on $0 \leqq x \leqq 1$ such that $\varphi(0) = k\varphi(1)$ with $\int_0^1 |\varphi'(x)|^2 dx < \infty$, harmonic functions on a connected region D and of summable square on D, harmonic functions of order 2 ($\Delta\Delta h = 0$) on a domain D such that $h = 0$ on the boundary and $\iint_D |\Delta h|^2 d\xi d\eta < \infty$. The space for which $N(x, y) = f(y - x)$, $f(x)$ on $-\infty < x < \infty$, a positive definite function as considered by Bochner, that is, $f(x) = \int_{-\infty}^{\infty} e^{ix\alpha} dV(\alpha)$, $V(\alpha)$ monotone bounded, is given [see Barnard and Goldstine, Bull. Amer. Math. Soc. **48**, 946–948 (1942); these Rev. **4**, 157]. *T. H. Hildebrandt.*

da Silva Dias, C. L. On the concept of an analytical functional. Anais Acad. Brasil. Ci. **15**, 1–9 (1943). (Portuguese) [MF 8744]

Let H be the "linear functional region" of all analytic functions regular on a closed set A of the complex sphere, and $F(y)$ a complex-valued functional on H. The author defines F to be "regular" if $F(y_n) \to F(y)$ whenever $y_n(t) \to y(t)$ uniformly on A and y_n, $y \varepsilon H$. The author shows that, if $F(y_1 + y_2) = F(y_1) + F(y_2)$ and $F(iy) = iF(y)$, then $F(\alpha y) = \alpha F(y)$ for any complex α and that for such functionals his definition of regularity is equivalent to Fantappiè's definition. Fantappiè's formula for linear functionals is also derived. *J. V. Wehausen* (Columbia, Mo.).

Gomes, Ruy Luís. Sur une généralisation de l'opérateur de projection $\mathcal{E}(I)$. Portugaliae Phys. **1**, 29–34 (1943). [MF 9195]

Let $\mathcal{E}(\lambda)$ denote resolution of the identity; let $\mathcal{E}(I) = \mathcal{E}(\lambda'') - \mathcal{E}(\lambda')$, where I is the interval $\lambda' \leqq \lambda \leqq \lambda''$ in R_1; let φ be an element of Hilbert space H; and let $U_\varphi(I) = \|\mathcal{E}(I)\varphi\|^2$. Then $U_\varphi(I)$ is an interval function over R_1 which serves to establish an exterior measure $U_\varphi^*(A)$ over R_1. Author's principal theorem: for each set $A \subset R_1$ there exists a uniquely determined projection operator $E(A)$ such that $U_\varphi^*(A) = \|E(A)\varphi\|^2$ for all $\varphi \varepsilon H$. A procedure is given for constructing $E(A)$ from $\mathcal{E}(I)$, and a few simple properties of U_φ^* are established. The author indicates how $U_\varphi^*(A)$ may represent the probability of obtaining, for values of a physical magnitude a in state φ, a point in the set A.
C. C. Torrance (Cleveland, Ohio).

Dunford, Nelson. Spectral theory. I. Convergence to projections. Trans. Amer. Math. Soc. **54**, 185–217 (1943). [MF 9102]

Dunford, Nelson. Spectral theory. Bull. Amer. Math. Soc. **49**, 637–651 (1943). [MF 9083]

To study convergence to projections the author considers a complex Banach space $\mathfrak{X}$ and a continuous linear operator T on the space. His goal is the determination of necessary and sufficient conditions for the convergence of a sequence of polynomials in T to a projection of the whole space on the manifold $\mathfrak{M}(P)$ consisting of all x such that $P(T)x = 0$, where P is a preassigned polynomial. There are, in general, three modes of convergence possible, uniform, weak and strong, and, when $\mathfrak{X}$ consists of measurable functions, also convergence almost everywhere. In carrying out the analysis the author has developed a theory paralleling the algebraic theory of matrices. In particular, generalizations of the minimal equation theorem, of Sylvester's result on the determinant of a matrix polynomial and of the decompositions of the space into a direct sum of subspaces determined by the spectrum are obtained and form the principal tools used in obtaining convergence theorems. Consider the class $\mathfrak{F}(T)$ of functions $f(\lambda)$ that are single-valued and regular in the closure of a domain D that contains the spectrum $\sigma(T)$ of T and is made up of a finite number of connected open sets whose closures are disjoint and whose boundary C is a finite number of disjoint closed rectifiable Jordan curves in the resolvent set of T. The function $f(T)$ is then

$$(1/2\pi i)\int_C f(\lambda) R_\lambda(T) d\lambda,$$

where $R_\lambda(T) = (\lambda I - T)^{-1}$. It is shown that, if $f_n(T)$ converges to a projection E with $E\mathfrak{X} \subset \mathfrak{M}(P)$, then either there are no $\lambda \varepsilon \sigma(T)$, where $f_n(\lambda) \to 0$, or else (1) the set σ of points $\lambda \varepsilon \sigma(T)$ where $f_n(\lambda) \to 1$ consists of a finite number of poles $\lambda_1, \cdots, \lambda_k$ of $R_\lambda(T)$; (2) if $\nu_1, \cdots, \nu_k$ are the orders of these poles, then $f_n(\lambda_i) \to 1$, $f_n^{(j)}(\lambda_i) \to 0$ ($j = 1, \cdots, \nu_i - 1$; $i = 1, \cdots, k$); (3) each λ_i is a root of $P(\lambda)$ of multiplicity not less than ν_i; (4) $\prod_{i=1}^k (\lambda_i I - T)^{\nu_i} E = 0$. If $P(\lambda) = \prod_{i=1}^k (\lambda - \lambda_i)^{\nu_i}$ and if f_n in $\mathfrak{F}(T)$ are such that $f_n(\lambda_i) \to 1$, $f_n^{(j)}(\lambda_i) \to 0$, $P(T)f_n(T) \to 0$, then the following are equivalent: (1) $f_n(T) \to E$, $E^2 = E$, $E\mathfrak{X} = \mathfrak{M}(P)$; (2) for each i, λ_i is in the resolvent set of T or is a pole of $R_\lambda(T)$; (3) X is the direct sum of $\mathfrak{M}(P)$ and the closed set $P(T)\mathfrak{X}$; (4) the set $(\lambda_i I - T)^{\nu_i + 1}\mathfrak{X}$ is closed. Finally ergodic theorems are obtained by replacing $P(T)$ by $I - T$ in the convergence results. *H. H. Goldstine* (Aberdeen, Md.).

Taylor, Angus E. Analysis in complex Banach spaces. Bull. Amer. Math. Soc. **49**, 652–669 (1943). [MF 9084]

In this address the author is principally concerned with a theory of analytic functions defined on a domain of the complex plane and having their values in a complex Banach space E. In the first section the notions of weak, strong and uniform regularity are discussed, it is shown that regularity is equivalent to weak regularity and that uniform regularity is equivalent to strong regularity on an open set. For spaces having a denumerable basis a theorem is obtained on the analyticity of the sum of a sequence of analytic functions. In this connection isolated singularities are discussed. The next two sections are concerned with the important notion of the resolvent R_λ of a linear transformation T of the space E into itself and with functions of such a transformation. The material here presented overlaps somewhat with that in Dunford [see the preceding review] but differs in the approach and point of view. In this paper the interest is primarily in the study of the singularities of the resolvent and of the projections $P = (-1/2\pi i)\int_C R_\lambda d\lambda$ which are of prime importance in an analysis of the regularity of the resolvent. As an illustration of the theory the author shows that the linear differential equation $d^n x/dt^n + a_1 d^{n-1}x/dt^{n-1} + \cdots + a_n x = y(t)$ with $x(0) = x'(0) = \cdots = x^{(n-1)}(0) = 0$ has as the solution

$$x = (-1/2\pi i)\int_C F^{-1}(\lambda)\lambda^n R_\lambda y d\lambda,$$

where $F(\lambda) = 1 + a_1\lambda + \cdots + a_n\lambda^n$ and C is a contour enclosing $\lambda = 0$ and excluding the zeros of $F(\lambda)$; he also gives a formula for the resolvent. The paper closes with a brief

survey of the work done on a theory of analytic functions whose domains and ranges are Banach spaces. It is indicated that one can start with a real Banach space B, form a complex Banach space by forming ordered pairs, define the norm $\|x+iy\|$ to be lub $(|f(x)|^2+|f(y)|^2)^{\frac{1}{2}}$ for all f in B^* of norm 1 and develop analogues of the Cauchy-Riemann equations. The converse problem of decomposing every complex Banach space into a real and imaginary part is considered briefly. *H. H. Goldstine* (Aberdeen, Md.).

Hamburger, H. L. Contributions to the theory of closed Hermitian transformations of deficiency-index (m, m). Quart. J. Math., Oxford Ser. **13**, 117–128 (1942). [MF 7636]

This paper is divided into three parts. In the first the author sketches the proofs of four theorems dealing with closed Hermitian transformations of deficiency-index (m, m) and their resolvents in Hilbert space $\mathfrak{H}$. The author promises to give detailed proofs in a later paper in which the case $m=\infty$, omitted in the present paper, will be considered. In the second part he defines a closed Hermitian transformation of deficiency (m, m) to be a prime transformation if $\mathfrak{H}$ does not contain any closed linear manifold which reduces H and with respect to which H is self-adjoint. He then proves the Theorem: in order that the spectrum of any closed Hermitian prime transformation H of deficiency-index (m, m) should contain only a finite number of points in every interval $a+\delta\leqq\lambda\leqq b-\delta$, it is necessary and sufficient that the m elements $\phi_s(\lambda+i\eta)$ should be continuous in the open interval $a<\lambda<b$, $|\eta|<\epsilon$, where $\{\phi_1(x), \phi_2(x), \cdots, \phi_m(x)\}$ is a set of m linearly independent characteristic solutions of the adjoint of H belonging to the characteristic value x. [The author notes that the necessity of this condition follows from a theorem of J. W. Calkin, Duke Math. J. **7**, 504–508 (1940); these Rev. **2**, 224.] In the third part the author proves two theorems concerning the construction of all closed Hermitian prime transformations of deficiency-index (m, m) to which there belongs a self-adjoint extension with a simple spectrum. It follows as a consequence of these two theorems that there exist closed Hermitian prime transformations of deficiency-index (m, m) whose self-adjoint extensions have spectra consisting of every point of the real axis. *J. Williamson* (Flushing, N. Y.).

Hille, Einar and Zorn, Max. Open additive semi-groups of complex numbers. Ann. of Math. (2) **44**, 554–561 (1943). [MF 8879]

The open connected additive semigroups of the complex plane are determined. Such groups consist of all vectors $\xi a+\eta b$, where a and b are suitably chosen vectors and where (i) the range of ξ is $\xi>0$ or $\xi<0$ or $-\infty<\xi<\infty$, (ii) $\eta>\phi(\xi)$, where $\phi(\xi)$ is an upper semicontinuous real function for which $\phi(\xi_1+\xi_2)\leqq\phi(\xi_1)+\phi(\xi_2)$ and $\underline{\lim}_{\xi\to 0}\phi(\xi)\leqq 0$. It is further established that for any open semigroup S there exists a Banach space E and a family of transformations T_s of E to E, $s\epsilon S$, such that T_s is a semigroup with parameter range S, T_s is holomorphic (in the strong topology) in S, and S is the maximal domain of analytic existence of T_s. The examples given to establish this fact are based on the definition $T_sf(z)=f(z+s)$, where $f(z)$ is a function analytic on S. *E. R. Lorch* (New York, N. Y.).

Halmos, Paul R. On automorphisms of compact groups. Bull. Amer. Math. Soc. **49**, 619–624 (1943). [MF 8864]

Let G be a compact Abelian group and let α be a continuous automorphism of G. Denoting by $f(x)$ a complex valued function over G, α is called ergodic if the only solutions f of the equation $f(\alpha x)=f(x)$ are constant almost everywhere. (This definition makes use of the existence of Haar measure over G; the uniqueness of Haar measure implies its invariance under every α.) Noting that α induces in a natural way an automorphism α^* of the discrete character group G^* of G, the author proves that α is ergodic if and only if α^* admits no finite orbits, that is, if no power of α^* admits fixed elements other than the identity. This and the group-theoretic result that α^* has an infinite number of orbits if it has no finite orbits (G^* being, in this theorem, any discrete group and α^* any automorphism of G^*) enable one to specify the "spectral type" of an ergodic α of G by giving the cardinal number of G, necessarily infinite. *P. A. Smith* (New York, N. Y.).

Theory of Probability

Mohr, Ernst. Bemerkung zum Buffonschen Nadelproblem. Z. Angew. Math. Mech. **22**, 170–171 (1942). [MF 8903]

Ghosh, Birendranath. On random distances between two rectangles. Science and Culture **8**, 464 (1943). [MF 9338]

Ghosh, Birendranath. On the distribution of random distances in a rectangle. Science and Culture **8**, 388 (1943). [MF 9339]

Chung, Kai Lai. On fundamental systems of probabilities of a finite number of events. Ann. Math. Statistics **14**, 123–133 (1943). [MF 8773]

This paper is concerned with certain linear combinations of probabilities of events and conjunctions of events of a given set. Some of these combinations can be interpreted as probabilities (for example, the probability that at least m of a set of n events will occur), whereas others are not interpretable but are merely auxiliary to the computation of probabilities. A system of numbers in one to one correspondence with a system of such linear combinations is said to form a fundamental system of probabilities provided these numbers determine all the probabilities associated with a given set of events. The author discusses several fundamental systems and obtains conditions under which the resulting probabilities are consistent. *A. H. Copeland.*

Chung, Kai Lai. Further results on probabilities of a finite number of events. Ann. Math. Statistics **14**, 234–237 (1943). [MF 9141]

This paper is concerned with certain linear combinations of the probabilities of a finite number of events. Some of the combinations are directly interpretable as probabilities whereas others are merely auxiliary to the computation of such probabilities. The author obtains equalities and inequalities relating these combinations. *A. H. Copeland* (Ann Arbor, Mich.).

Wolfowitz, J. On the theory of runs with some applications to quality control. Ann. Math. Statistics **14**, 280–288 (1943). [MF 9145]

The author presents a survey of the development of the theory of probability as related to runs and also of the applications of this theory to problems in quality control.

If the elements of an ordered set be assigned to various categories, then runs in which one or more consecutive elements fall into the same category will occur within the ordered set. The assignment to categories can be accomplished in many ways in accordance with various characteristics of the observations and the same can be said for the ordering of the set. Thus the elements may be assigned to one of two categories according as the observations exceed or fail to exceed the medium (run above or below the median) or according as they exceed or fail to exceed their immediate predecessors (run up or down). The ordering may be that in which the observations are made. Again the observations may be obtained from two distinct populations (two machines or two workers) constituting two categories and the ordering in accordance with the magnitude of the observations. In this latter case long runs and hence few runs indicate a discrepancy between the distribution functions of the two populations. The author gives a number of such examples and concludes with an indication as to directions in which research in this field can profitably be pursued. A useful bibliography is appended.

A. H. Copeland (Ann Arbor, Mich.).

Bernstein, S. N. Retour au problème de l'évaluation de l'approximation de la formule limite de Laplace. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 3–16 (1943). (Russian. French summary) [MF 8525]

Let X_n be a variable distributed according to the Bernoulli law $(p+q)^{(n)}$. Suppose that $npq \geqq 62.5$ and that $m_0 < m_1$ are positive integers. It is shown that

$$\text{(1)} \qquad \text{Prob}\,\{m_0 \leqq X_n < m_1\} > \pi^{-\frac{1}{2}} \int_{a_0}^{a_1} e^{-z^2} dz,$$

where a_0 and a_1 are defined by

$$\text{(2)} \qquad m_i + \tfrac{1}{2} = np + a_i(2npq)^{\frac{1}{2}} + (q-p)a_i^2/3.$$

The inequality (1) is reversed if the term $\frac{1}{2}$ on the left of (2) is replaced by $-\frac{1}{2}$, provided that the resulting a_0 is positive and $a_1 \leqq (2npq)^{\frac{1}{2}}$. Several inequalities of a similar character follow. *W. Feller* (Providence, R. I.).

Gnedenko, B. Sur la distribution limite du terme maximum d'une série aléatoire. Ann. of Math. (2) **44**, 423–453 (1943). [MF 8873]

Let $x_1, x_2, \cdots$ be a sequence of independent random variables all having the same distribution function $F(x)$ and let $\xi_n = \max(x_1, \cdots, x_n)$. Obviously, the distribution function of ξ_n is $F_n(x) = \{F(x)\}^n$. The present paper is devoted to the study of $\lim_{n\to\infty} F_n(a_n x + b_n)$, where $\{a_n\}$ $(a_n > 0)$ and $\{b_n\}$ are sequences of constants. Many results in this direction were known [cf. M. Fréchet, Ann. Soc. Polon. Math. **6**, 93–96 (1927); R. A. Fisher and L. H. C. Tippett, Proc. Cambridge Philos. Soc. **24**, part II, pp. 180–190 (1928); R. de Mises, Rev. Math. Union Interbalkan. **1**, 141–160 (1939)]. In particular, it was known [Fisher and Tippett, loc. cit.] that the only limiting distribution functions which may be obtained by a suitable choice of F and the sequences $\{a_n\}$ and $\{b_n\}$ are of the types $\Phi_\alpha(x)$, $\Psi_\alpha(x)$ and $\Lambda(x)$, where

$$\Phi_\alpha(x) = \begin{cases} 0, & x \leqq 0, \\ \exp(-x^{-\alpha}), & x > 0, \end{cases} \qquad \Psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha), & x \leqq 0, \\ 0, & x > 0, \end{cases}$$

and $\Lambda(x) = \exp(-e^{-x})$. (A distribution function is said to be of type Φ if it can be put in the form $\Phi\,(ax+b)$.) The author (who, incidentally, proved the above theorem independently) refines this result in various ways. The following theorem is indicative of the type of results obtained. In order that sequences $\{a_n\}$ $(a_n > 0)$ and $\{b_n\}$ can be chosen so that $\lim_{n\to\infty} F_n(a_n x + b_n)$ is of type Φ_α it is necessary and sufficient that, for every $k > 0$,

$$\lim_{x\to\infty} \frac{1-F(x)}{1-F(kx)} = k^\alpha.$$

M. Kac (Ithaca, N. Y.).

Forsythe, G. E. Cesàro summability of independent random variables. Duke Math. J. **10**, 397–428 (1943). [MF 8476]

The author studies generalizations of the weak law of large numbers and the central limit theorem of the calculus of probability, resulting from replacing C_1 by more general Cesàro methods of summability (C_r). In generalizing the central limit theorem the author restricts himself to the so-called "normal families" of random variables. In this case it is shown that, for $s > r \geqq 1$, summability C_s is nontrivially stronger than C_r. The problem of comparing the C_r methods for $\frac{1}{2} \leqq r < 1$ is still open. Many results extending the weak law of large numbers are given. It is pointed out that the case $0 < r < 1$ offers great difficulties. *M. Kac.*

Mathematical Statistics

★Wilks, S. S. Mathematical Statistics. Princeton University Press, Princeton, N. J., 1943. xi+284 pp. $3.75.

In a sense the book under review is unique. The great number of books on the market dealing with statistics and probability is notorious. All these fall roughly into two categories. The majority describe "statistical methods," frequently ignoring the basic ideas and the mathematics behind these methods, occasionally misinterpreting them. The books of the other category, some of them excellent [Borel Series, Cramér, Kolmogoroff, Uspensky], deal with mathematical theory of probability with only occasional glimpses on some particular questions pertaining to statistical theory. In contrast, the aim of Wilks is to give the students an organized source of information concerning the theory of statistics as such. The need of such a source is great and the book under review deserves general recognition. The book, covering the author's courses at Princeton University, was compiled with the collaboration of H. Scheffé, T. W. Anderson and D. F. Votaw. It contains eleven chapters as follows: (1) Introduction; (2) Distribution functions; (3) Some special distributions; (4) Sampling theory; (5) Sampling from a normal population; (6) On the theory of statistical estimation; (7) Tests of statistical hypotheses; (8) Normal regression theory; (9) Application of normal regression theory to analysis of variance problems; (10) On combinatorial statistical theory; (11) An introduction to multivariate statistical analysis.

Chapter (II), covering 42 pages, gives a somewhat hasty sketch of the mathematical theory of distribution functions, including the integral of Stieltjes, the moment generating functions and the problem of moments. The reviewer is inclined to think that this chapter is less successful than the others, mostly for reasons beyond the author's control. The students interested in statistics are usually ignorant of various sections of analysis which are prerequisite to the theory of statistics. Hence the dilemma arises as to whether to begin with an introduction to the theory of functions

(with a risk of not being able ever to embark on the theory of statistics itself) or to commence with an enumeration of results in the analysis which are to be used later (with the risk that the students will misunderstand these results and then will fail to follow the main topic). Failing an adjustment in the curriculum in pure mathematics, some sort of compromise between these two extremes appears to be unavoidable and, almost invariably, the corresponding chapter is sketchy and superficial. It seems that Chapter (II) did not entirely escape the dangers of this situation.

Chapters (III), (IV) and (V) are given to the study of the most important particular distributions, first of a single variable and then of several variables and of various functions of them. Here a student dissatisfied with the mystery of various tables reproduced and used in the popular books on "statistical methods" will find all the information he needs. Beginning with the binomial, multinomial and the Poisson laws, the author proceeds to deduce and to discuss the Pearson curves, the distribution of means (including the limiting case) of the sample S.D. of χ^2, of Student's t, of Snedecor's F and of the sample correlation coefficient. The next chapters, (VI) and (VII), are given to what may be considered as the theory of statistics proper, namely, to the theories of estimation and of testing statistical hypotheses. The idea of confidence intervals is explained and illustrated in several important examples. The author also discusses single estimates and their classification. However, the important problem of a rational choice between systems of confidence intervals is not treated in detail.

In chapter (VII) the author explains the λ criterion for deducing statistical tests and briefly discusses the conception of the power function of a test. One feels inclined to compliment the author on the omission of anything relating to "fiducial distributions," a fertile source of various misunderstandings. Chapters (VIII) and (IX) deal with the λ tests applied to hypotheses specifying numerical values of several linear combinations of population means of n mutually independent normal variables, as first treated by Kolodziejczyk. Introductory simple examples are followed by the general case and this then is applied to deduce Fisher's analysis of variance. The important experimental designs of randomized blocks, Latin squares and Graeco-Latin squares supply useful illustrations. In connection with these chapters one is led to regret the omission of the discussion of the tables of probabilities of errors of the second kind, which are basic for a rational design of experiments.

Chapter (X) gives an interesting outline of the recent results concerned with "runs" and "matching," followed by the description of sampling inspection methods by Dodge and Romig. The last chapter (XI) compiles the results of the author himself, of Fisher, Hotelling, Hsu, Wishart and others, all concerned with multivariate normal populations and connected with the λ tests of various statistical hypotheses.

The book as a whole is a very useful one. As mentioned, it is as yet the only book where a student can get information on the theories of testing hypotheses and on estimation. Besides, it contains a wealth of information on the distributions of various important statistics. Actually distributions of statistics seems to be the main "center of gravity" of the book with the λ tests of hypotheses being a secondary one. The references in the text seem to be somewhat uneven. Occasionally some secondary result is credited to its author by name [for example, on page 135 the reviewer gets credit for a result of obviously second rate importance]. On the other hand some bigger sections of the theory, representing the results of difficult research, are given without a direct credit to the authors. For example, the exposition of properties of the maximum likelihood estimates does not include any reference either to Doob or to Hotelling who were the first to prove them. However, the list of literature for supplementary reading at the end of the book is quite extensive. The book would probably gain if it were supplemented by a few more examples, particularly with a few numerical examples. However, as it stands, it is an excellent addition to modern statistical literature. *J. Neyman.*

Hirschman, Albert O. On measures of dispersion for a finite distribution. J. Amer. Statist. Assoc. **38**, 346–352 (1943). [MF 9049]

The author obtains a number of relations among the means, medians, variances and mean deviations from the means and medians of an even-numbered sample and of the four subsamples formed from it by the elements less than and greater than the mean and the elements less than and greater than the median, respectively. *J. Wolfowitz.*

Cowden, Dudley J. Correlation concepts and the Doolittle method. J. Amer. Statist. Assoc. **38**, 327–334 (1943). [MF 9048]

Platt, John R. A mechanical determination of correlation coefficients and standard deviations. J. Amer. Statist. Assoc. **38**, 311–318 (1943). [MF 9046]

Hayes, Samuel P., Jr. Tables of the standard error of tetrachoric correlation coefficient. Psychometrika **8**, 193–203 (1943). [MF 9067]

Herein are tabulated values of the full expression given by Pearson for the standard error of tetrachoric k. It is observed that the shortened approximation later given also by Pearson, which has come into common use, often differs considerably in value from that given by the longer formula but it not remarked that even the latter is itself an approximation presumably valid only for not too small samples.
C. C. Craig (Ann Arbor, Mich.).

Bose, Purnendu. On the reduction formulae for the incomplete probability integral of the multiple correlation coefficient of the second kind. Science and Culture **7**, 171–172 (1941). [MF 9288]

Bose, Purnendu. Certain moment calculations connected with multivariate normal populations. Science and Culture **7**, 411–412 (1942). [MF 9290]

Shrivastava, M. P. The distribution of the mean for certain Bessel function populations. Science and Culture **6**, 244–245 (1940). [MF 9285]

Radhakrishna Rao, C. On the sum of n observations from different gamma type populations. Science and Culture **7**, 614–615 (1942). [MF 9292]

Hsu, P. L. The limiting distribution of a general class of statistics. Acad. Sinica Science Record **1**, 37–41 (1942). [MF 8831]

The author states, without detailed proofs, three theorems concerning the limiting distribution, as the sample size becomes infinite, of a general function $f(\bar{u}_1, \cdots, \bar{u}_m)$ of the means $\bar{u}_1, \cdots, \bar{u}_m$ in a sample from an m-variate universe,

in which, of course, f satisfies certain regularity conditions. In the first theorem it is assumed that a bilinear form in the first partial derivatives of f valued at universe means σ^2 is positive, and it is stated that f less its value at universe means the quantity divided by $n^{-\frac{1}{2}}\sigma$ tends to be a normally distributed standard variable. In the second and third theorems σ^2 is assumed zero but it is given that the matrix of second partial derivatives of f valued at expected values is definite. In this case the limiting distribution of n ($f-\bar{f}$ valued at expected values) becomes that of a linear form in the squares of standardized normally distributed variables with the positive latent roots of the matrix as coefficients. The familiar χ^2-distribution is a special case.

C. C. Craig (Ann Arbor, Mich.).

Hsu, P. L. Some simple facts about the separation of degrees of freedom in factorial experiments. Sankhyā 6, 253–254 (1943). [MF 9077]

Denote by Δ_k an orthogonal k-rowed matrix whose elements in the first row are all $k^{-\frac{1}{2}}$. Let $x_{\alpha\beta}$ ($\alpha=1, \cdots, k$; $\beta=1, \cdots, l$) be the measurements in a two factor experiment at k and l levels, respectively. Let $\|y_{\alpha\beta}\|=\Delta_k\|x_{\alpha\beta}\|\Delta_l'$, where Δ_l' is the transposed matrix to Δ_l. The author shows that main effects and interactions can be conveniently and concisely expressed by the $y_{\alpha\beta}$. The method can be extended to any number of factors. The formulae for a 2, 3 and 4 factor experiment are given explicitly in the paper.

H. B. Mann (Barrytown, N. Y.).

Dodd, Edward L. A transformation of Tippett random sampling numbers into numbers normally distributed. Bol. Mat. 15, 73–77 (1942). [MF 8771]

Tippett's random sampling numbers are 4-digit numbers which may be considered as independent observations (without replacement) from the population of all 4-digit numbers, where each 4-digit number has the probability 10^{-4}. The author describes a method of transforming these numbers into numbers which may be considered as independent observations from a normal population with zero mean and unit variance. The transformation is carried out as follows. Each Tippett number is considered with a decimal point prefixed and a 5 affixed. For any such Tippett number y the corresponding transformed number x is the solution of the equation $y=F(x)$, where

$$F(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}e^{-t^2/2}dt.$$

The transformed values of the first 800 Tippett numbers are given in a table.

A. Wald (New York, N. Y.).

Chandra Sekar, C. A note on the inverse sine transformation. Sankhyā 6, 195–198 (1942). [MF 8640]

The analysis of variance tests in current use are based on the assumption that the distribution of the random variable x under consideration is normal. If x has the binomial distribution, that is, if x is the number of successes in a sequence of n independent trials, the difficulties arising from the nonnormality of x can be overcome to a certain extent by using the transformation $y=\arcsin (x/n)^{-\frac{1}{2}}$ or the slightly different transformation $z=\arcsin \{(x\pm\frac{1}{2})/n\}^{\frac{1}{2}}$, where $\frac{1}{2}$ is added or removed from x as x is less or greater than $\frac{1}{2}n$. In this paper the author studies the relative advantages of these transformations when n is small. The mean, variance and first Pearsonian coefficient β_1 are computed for x, y and z when $n=15$ and $q=.2, .3, .4, .5$ (q is the parameter of the binomial distribution). The author finds that these values alone do not indicate any definite advantage of z over y. Further studies are based on hypothetical experiments constructed with the help of Tippett's random numbers. The experimental results seem to indicate that the test based on the z-transformation will reject the null hypothesis (when it is true) slightly less frequently than the test based on the y-transformation.

A. Wald (New York, N. Y.).

Ganguli, Mohonlal. A method of estimating variance of sample grand-mean and zone variances in unequal nested sampling. Science and Culture 6, 724 (1941). [MF 9287]

Roy, S. N. and Banerjee, Kalishankar. On hierarchical sampling, hierarchical variances and their connexion with other aspects of statistical theory. Science and Culture 6, 189 (1940). [MF 9284]

Festinger, Leon. An exact test of significance for means of samples drawn from populations with an exponential frequency distribution. Psychometrika 8, 153–160 (1943). [MF 9064]

A random variable X is usually said to have the exponential distribution if $2(X-\beta)/\sigma$ has the χ^2-distribution with two degrees of freedom. Tests have been advanced for various hypotheses about populations with exponential distributions, and the properties of such tests have been studied; [see, for example, Paulson, Ann. Math. Statistics 12, 301–306 (1941); these Rev. 3, 174 and references there]. The present author restricts himself to the case $\beta=0$, so that there is only one unknown parameter σ, which then becomes the mean of X. He proposes the application of this case to certain psychological problems where J-shaped distributions are encountered. Let $\bar{x}$ be the mean of a sample of size n. The fact that $2n\bar{x}/\sigma$ has the χ^2-distribution with $2n$ degrees of freedom (additive property of the χ^2-law) is employed to obtain confidence intervals for σ, and the fact that the quotient of two independent χ^2-variables has the F-distribution, to obtain a test for the equality of the means of two such distributions. In every case the confidence coefficient $1-\alpha$ or the significance level α is calculated incorrectly: the correct α is always twice the value stated.

H. Scheffé (Princeton, N. J.).

Cochran, W. G. The comparison of different scales of measurement for experimental results. Ann. Math. Statistics 14, 205–216 (1943). [MF 9138]

This paper treats the analysis of data obtained when the results of a replicated experiment are measured on different scales. The distribution of the observations in the several scales is assumed jointly normal with constant covariance matrix. Tests of the following hypotheses are discussed in detail: (a) equality of scales, (b) constant differences between scales, (c) linear relationships between pairs of scales. When $p=2$, tests of (a) and (b) are reduced to classical analysis of variance. When $p>2$, the author, by an ingenious artifice, reduces the problem to the Wilks-Lawley test, for which a criterion is available in the literature. It is shown that this test is equivalent to testing whether certain canonical correlation coefficients are zero. The problem of testing (c) is proved to be identical with Fisher's test of "collinearity." It is shown that, for $p=2$, the least squares method leads to the same criterion. A measure of the sensitivity of each scale, which may be very roughly described as the ratio of the variance among the

population treatment means to the scale error variance, is proposed. The problem of estimating the sensitivity is discussed. *J. Wolfowitz* (New York, N. Y.).

v. Mises, R. On the problem of testing hypotheses. Ann. Math. Statistics 14, 238–252 (1943). [MF 9142]

Investigations into the problem of testing hypotheses are pursued in a direction now generally abandoned by workers in the field. This direction might be described by the title of an early paper by Neyman and Pearson: "The testing of statistical hypotheses in relation to probabilities a priori" [Proc. Cambridge Philos. Soc. 29, 492–510 (1933)]. Let ϑ be the unknown parameter and $P(x|\vartheta)$ be the (cumulative) distribution function of the sample X, where both X and ϑ may be vectors. As usual the functional form of $P(x|\vartheta)$ is assumed known; however, it is required further that we consider an unknown distribution $P_0(\vartheta)$ of the parameter. By employing the concept of the two types of error possible in applying a test, a formula analogous to that of Neyman and Pearson [ibid., p. 495] is obtained for the total error chance P_E of a test; P_E depends on $P_0(\vartheta)$. The success rate S is defined as $1-\sup P_E$ for $P_0(\vartheta)$ in an admissible class $\{P_0(\vartheta)\}$. The author considers using S as the criterion for the comparison of tests, but does not explicitly recommend it. Some of his results are the following. (1) For both simple and composite hypotheses, finding the test with highest S becomes a "simple maximum-minimum problem." (2) Under certain restrictions no test can have an $S>\frac{1}{2}$. (3) If, however, $P_0(\vartheta)$ is independent of the sample size n, then under certain conditions tests can be found such that, for any $\epsilon>0$, $S>1-\epsilon$ for n sufficiently large. The author mentions that the test with highest S coincides with the "best" test in Wald's general theory employing a given weight function [On the Principles of Statistical Inference, Notre Dame Mathematical Lectures, no. 1, University of Notre Dame, Notre Dame, Ind., 1942; these Rev. 4, 25] in the special case where the weight function can take on only the values 0 and 1, but claims the philosophy is different, his approach being "independent of any arbitrary assumption about an error weight." The reviewer presumes to add that, while this theory flows simply and elegantly from the criterion of success rate, the desirability of this criterion is questionable from the standpoint of practice. Thus, to test a simple hypothesis $\vartheta=\vartheta_0$ we must assume a class $\{P_0(\vartheta)\}$ of parameter distributions; in any case in which we are willing to assume further that all $P_0(\vartheta)$ are continuous at ϑ_0, we can always get a test with $S=1$ by invariably rejecting the hypothesis. Also questionable is the dogmatic assertion ". . . it would be senseless to speak of error chances without assuming an over-all distribution $P_0(\vartheta)$ exists." *H. Scheffé* (Princeton, N. J.).

Neyman, J. Basic ideas and some recent results of the theory of testing statistical hypotheses. J. Roy. Statist. Soc. (N.S.) 105, 292–327 (1942). [MF 8624]

In this paper an exposition is given of the basic ideas of the modern theory of testing statistical hypotheses as developed by the author and E. Pearson. An essential feature of this theory is that test criteria are developed which do not depend on the knowledge or existence of an a priori probability distribution of admissible hypotheses. This is of great importance, since the a priori probability distribution is usually unknown, and in some cases even its existence cannot be postulated. In the Neyman-Pearson theory the selection of proper test criteria is based on the notion of errors of the first and second kind. An error of the first kind is committed if the hypothesis H_0 which is to be tested is rejected when it is true, and an error of the second kind is committed if H_0 is accepted when some alternative hypothesis $H_1\neq H_0$ is true. The fundamental idea underlying the Neyman-Pearson theory can be briefly stated as follows: among the test procedures for which the probability of an error of the first kind has the same value that one is to be preferred for which the probability of an error of the second kind is a minimum. If a test procedure exists for which the probability of an error of the second kind is minimized simultaneously for all admissible alternative hypotheses, it provides an ideal solution of the problem and is called a uniformly most powerful test. If a uniformly most powerful test does not exist, additional criteria are introduced for the selection of the test procedure to be used. In addition to an outline of the general theory, particular problems and results are discussed which are related to, and are good illustrations of, the theory. *A. Wald* (New York, N. Y.).

Scheffé, Henry. On a measure problem arising in the theory of non-parametric tests. Ann. Math. Statistics 14, 227–233 (1943). [MF 9140]

Tests of statistical hypotheses where the forms of the distribution functions of the variates are unknown (except perhaps for continuity, etc.) are called nonparametric tests. The nonparametric tests discussed in the literature are either invariant under topological transformation (such as functions of runs or of ranks) or else employ the Fisher-Pitman idea of considering only the permutations of the values actually observed. In its full generality the latter method includes the former. Both methods yield tests of exact size, that is, the probability of rejecting the hypothesis being tested is a known constant independent of the unknown distribution functions when the hypothesis being tested is true.

The author formulates in a rigorous manner the requirements of exact size in the nonparametric case and proves, among others, the interesting result that the general Fisher-Pitman tests are the most general nonparametric tests whose size is exact. To put it precisely, a Borel set s in a k-dimensional Euclidean space is said to have the structure S if, for every point E with no two coördinates equal, a fixed number M $(0<M<k!)$ of points of the set p obtained by permuting the coördinates of E are in the set s. In order that the probability of the Borel set w be a constant α $(0<\alpha<1)$ for all continuous probability density functions

$$f_k(x_1, \cdots, x_k)=\prod_{j=1}^{k} f(x_j),$$

it is necessary that w has the structure S. *J. Wolfowitz* (New York, N. Y.).

Nair, K. Raghavan and Radhakrishna Rao, C. Confounded designs for asymmetrical factorial experiments. Science and Culture 6, 313–314 (1941). [MF 9289]

Nair, K. Raghavan and Radhakrishna Rao, C. A note on partially balanced incomplete block designs. Science and Culture 7, 568–569 (1942). [MF 9291]

Nair, K. Raghavan and Radhakrishna Rao, C. Incomplete block designs for experiments involving several groups of varieties. Science and Culture 7, 615–616 (1942). [MF 9293]

Mathematical Biology

Kostitzin, V. A. Evolution. Revista Acad. Colombiana Ci. Exact. Fis. Nat. **5**, 241–245 (1942). (Spanish) [MF 9098]

Rashevsky, N. Note on the Hamiltonian principle in biology and in physics. Bull. Math. Biophys. **5**, 65–68 (1943). [MF 8414]

According to Lotka, the course of events in the biological world is determined by maximizing the total energy exchange between different biological units. This the author interprets to mean that there exists a "Hamiltonian" $F(x_1, \cdots, x_n, x_1', \cdots, x_n')$ for which always $\delta\int F dt = 0$. This F has dimension ml^2t^{-3}, while the original Hamilton function has dimension ml^2t^{-2}. The author suggests that there may exist a universal law of nature

$$\delta\int(\alpha F - \beta H)dt = 0.$$

"We may make the hypothesis that α and β are two universal constants. All systems for which $\alpha F \gg \beta H$ we may define as biological; all systems for which $\alpha F \ll \beta H$ we may define as physical. When αF is comparable to βH we have borderline systems. Such may be perhaps the case of single cells." *W. Feller* (Providence, R. I.).

Rashevsky, N. Mathematical biophysics of cell division. Bull. Math. Biophys. **5**, 99–102 (1943). [MF 9061]

In the paper reviewed above the author proposed generalizing Hamilton's principle for biological systems by including in the integrand function a term τF, along with the usual Hamiltonian, τ being a time constant and F representing the total energy flow. Frictional losses are then accounted for by following Whittaker's procedure [A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, The University Press, Cambridge, England, 1927]. This paper applies the principle to cellular elongation and compares the results with the equations previously derived on the basis of the mechanical effects of diffusion forces. The principal difference lies in the fact that, whereas the mechanical theory predicted no elongation for large diffusion coefficient in the external medium, with the present theory this is not the case. *A. S. Householder* (Chicago, Ill.).

Rashevsky, N. On the form of plants and animals. Bull. Math. Biophys. **5**, 69–73 (1943). [MF 8415]

With reference to his previous paper [Bull. Math. Biophys. **5**, 33–47 (1943)] the author analyzes the possible forms of trees by considering the branches as cantilever beams and the trunk as a compressed column. He assumes an exponential relation between the compressive strength and the density and a similar relation for the maximum deflection at rupture. The modulus of elasticity is taken proportional to the density. These assumptions are supplemented by two intuitive relations concerning the amount of metabolic flow. *I. Opatowski* (Chicago, Ill.).

Pitts, Walter. The linear theory of neuron networks: the dynamic problem. Bull. Math. Biophys. **5**, 23–31 (1943). [MF 8141]

A continuation of the author's work on neuron networks, the nerve cells idealized to give linear response between a threshold and saturation point [Bull. Math. Biophys. **4**, 169–175 (1942); these Rev. **4**, 202]. Here equations are derived for the response as a function of time in terms of matrices characterizing the network topology and the metrical properties of the neurons. Conditions are given for stability of a network and for the existence of a network with given characteristics. A canonical form of networks is also developed. *C. E. Shannon* (New York, N. Y.).

TOPOLOGY

Alexandroff, A. D. On the extension of a Hausdorff space to an H-closed space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 118–121 (1942). [MF 8654]

A Hausdorff space is H-closed if it is closed in any Hausdorff space containing the given one as a subspace. The author gives a new proof of the theorem that any Hausdorff space R can be extended to an H-closed space. (This implies that R is everywhere dense in the extension.) Earlier proofs are due to Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)], Tychonoff [Math. Ann. **102**, 544–561 (1929)] and Fomin [Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 285–294 (1940); these Rev. **2**, 320]. The proof is accomplished by means of transfinite induction in the definition of a well-ordered set of spaces, beginning with R, each of which contains all earlier ones, and is carried out with the aid of four lemmas, of which the concluding and conclusive one is lemma D: the power of a Hausdorff space containing an everywhere dense subset of power m does not exceed 2^{2^m}. This lemma is attributed to Pospíšil [Ann. of Math. (2) **38**, 845–846 (1937)], but a proof is included. The considerations of this paper are shown to yield theorems analogous to the main theorem. For example, "an arbitrary topological group can be extended to an absolutely closed group, that is, such that it is a closed subset of any group in which it is contained as a subgroup and a subspace." *A. B. Brown.*

Fomin, S. Extensions of topological spaces. Ann. of Math. (2) **44**, 471–480 (1943). [MF 8875]

A detailed exposition of an earlier note [Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 285–294 (1940); these Rev. **2**, 320]. *S. Eilenberg* (Ann Arbor, Mich.).

Shanin, N. A. On special extensions of topological spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 6–9 (1943). [MF 8692]

If $\mathfrak{A}$ is any family of sets, $\Delta\mathfrak{A}$ and $\nabla\mathfrak{A}$ denote the families of all intersections and sums of finite sets of elements of $\mathfrak{A}$. If E is a T-space (that is, E and the vacuous set are closed, the sum of a finite set and the intersection of an arbitrary set of closed sets is closed), a family of closed sets $\mathfrak{B}$ is a closed basis if every closed set is the intersection of elements of $\mathfrak{B}$; $\mathfrak{B}$ is reduced if it contains the vacuous set and $\nabla\mathfrak{B} = \mathfrak{B}$. Let $R \subset E$. A family $\mathfrak{F}$ of closed sets of R is called a closed framework of E if the closures in E of the elements of $\mathfrak{F}$ is a closed basis of E. If R is any T-space and $\mathfrak{B}$ is any reduced closed basis of R, there is an $(\omega, \mathfrak{B})$-extension of R and this is unique up to a topological mapping leaving invariant every point of R, that is, there is a bicompact T-space $S \supset R$ such that $\mathfrak{B}$ is a closed framework of S in R, every point of $S - R$ is a closed set, and for any finite system $[F_i]$ of $\mathfrak{B}$

$$\overline{\sum F_i} = \sum \overline{F_i}.$$

A nonvacuous family of closed sets is called a χ-family if the intersection of any finite number of interiors is nonvacuous. A set is quasi-bicompact if every χ-family has a common point. For any T-space R and any reduced closed basis $\mathfrak{B}$ there is a $(\chi, \mathfrak{B})$-extension of R which is unique up to a homeomorphism leaving invariant every point of R; that is, there is a quasi-bicompact space $S \supset R$ such that $\mathfrak{B}$ is a closed framework of S in R, every point of $S-R$ is closed, and if $F \epsilon \Delta\mathfrak{B}$ and interior F(rel R) is vacuous then F is closed in S, for any finite system $[F_i]$ of $\mathfrak{B}$, $\overline{\sum F_i} = \sum \overline{F}_i$. Any $(\omega, \mathfrak{B})$-extension is a $(\chi, \mathfrak{B})$-extension if and only if any $P \epsilon \Delta\mathfrak{B}$ and interior P(rel R) vacuous is bicompact in R.
W. L. Ayres (Lafayette, Ind.).

Shanin, N. A. On separation in topological spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 110–113 (1943). [MF 8686]

If E is a set and $\mathfrak{F}$ is a family of subsets of E, $E\{\mathfrak{F}\}$ denotes the family of the complements of the elements of $\mathfrak{F}$ in E. A family $\mathfrak{P}$ will be said to be conjugate to a family $\mathfrak{R}$ if for $P \epsilon \mathfrak{P}$, $R \epsilon \mathfrak{R}$, $P \subset R$ there exist $P' \epsilon \mathfrak{P}$, $R' \epsilon \mathfrak{R}$ so that $P \subset R' \subset P' \subset R$. A family $\mathfrak{F}$ of closed sets of a T-space E will be normal if $\mathfrak{F}$ and $E\{\mathfrak{F}\}$ are conjugate; $\mathfrak{F}$ is weakly regular if for $F \epsilon \mathfrak{F}$ and $x \epsilon E-F$ there is $G \epsilon E\{\mathfrak{F}\}$ so that $F \subset G \subset E-x$; $\mathfrak{F}$ is regular if for $F \epsilon \mathfrak{F}$ and $x \epsilon E-F$ there exist $G \epsilon E\{\mathfrak{F}\}$ and $F' \epsilon \mathfrak{F}$ so that $F \subset G \subset F' \subset E-x$. A pair of sets F, G will be canonical with respect to $\mathfrak{F}$ if $F \subset G$ and there are conjugate families $\mathfrak{P}$, $\mathfrak{R}$ so that $F \epsilon \mathfrak{P} \subset \mathfrak{F}$, $G \epsilon \mathfrak{R} \subset E\{\mathfrak{F}\}$; $\mathfrak{F}$ will be completely regular if for $F \epsilon \mathfrak{F}$ and $x \epsilon E-F$ there is a set $G \epsilon E\{\mathfrak{F}\}$ such that $F \subset G \subset E-x$ and F, G are canonical in the family $\mathfrak{F}$. A T-space E will be called normal, weakly regular, etc., according to whether its family of closed sets is normal, weakly regular, etc.

Let $\mathfrak{B}$ be a reduced closed basis of the T-space R and S be the $(\omega, \mathfrak{B})$-extension of R. In order that S be (a) a normal T-space, (b) a weakly regular T-space, (c) a regular T-space, (d) a completely regular T-space, (e) a T_0-space, (f) a T_1-space or (g) a T_2-space, it is necessary and sufficient that (a) $\Delta\mathfrak{B}$ be a normal family, (b) $\Delta\mathfrak{B}$ be a weakly regular family, (c) $\Delta\mathfrak{B}$ be a regular normal family, (d) $\Delta\mathfrak{B}$ be a completely regular normal family, (e) R be a T_0-space, (f) R be a T_1-space and $\Delta\mathfrak{B}$ be a weakly regular family, (g) R be a T_2-space and $\Delta\mathfrak{B}$ be a weakly regular normal family. By the weight of a space is meant the minimum cardinal number of the closed bases of the space. It is shown that every normal T-space (completely regular T-space) may be extended to a bicompact normal T-space (completely regular T-space) of the same weight. These results were proved by Tychonoff for T_1-spaces [Math. Ann. **102**, 544–561 (1929)]. The $(\chi, \mathfrak{B})$-extension of a T-space R is a T_0-space if and only if R is a T_0-space. The extension is a T_1-space if and only if R is a T_1-space and for $F \epsilon \Delta\mathfrak{B}$, $x \epsilon R-F$ there exists $G \epsilon R\{\Delta\mathfrak{B}\}$ so that interior F(rel R) $\subset$ closure of $G \subset R-x$.
W. L. Ayres (Lafayette, Ind.).

Shanin, N. A. On the theory of bicompact extensions of topological spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 154–156 (1943). [MF 8680]

If $\mathfrak{F}$ and $\mathfrak{G}$ are families of closed sets, $\mathfrak{G}$ is said to be a closed basis of $\mathfrak{F}$ if every element of $\mathfrak{F}$ can be represented as the intersection of elements of $\mathfrak{G}$. The closed basis $\mathfrak{G}$ of $\mathfrak{F}$ is called an accurate closed basis if for $F \epsilon \mathfrak{F}$, $F' \epsilon \mathfrak{F}$, $FF'=0$ there exist $G \epsilon \mathfrak{G}$ and $F'' \epsilon \mathfrak{F}$ so that $F' \subset G \subset F''$ and $FF''=0$; $\mathfrak{F}$ is subject to $\mathfrak{G}$ if for any finite system $[F_i]$ of $\mathfrak{F}$ and $G \epsilon \mathfrak{G}$, $H \epsilon E\{\mathfrak{G}\}$ so that $\prod F_i \subset G \subset H$ there exists a system $[G_i]$ of $\mathfrak{G}$ with each $F_i \subset G_i$ and $\prod G_i \subset H$. The author then states the following uniqueness theorem. Let $\mathfrak{B}$ and $\mathfrak{C}$ be reduced closed bases of the T-space R and let S be the $(\omega, \mathfrak{B})$-extension of R. In order that S be the $(\omega, \mathfrak{C})$-extension of R it is necessary and sufficient that (1) $\Delta\mathfrak{C}$ be an accurate closed basis of $\Delta\mathfrak{B}$ and (2) the family $\Delta\mathfrak{C}$ be subject to $\Delta\mathfrak{B}$. The Čech-extension of a completely regular T-space is a Wallman-extension if and only if R is a normal T-space. The bicompact completely regular T-space S is called a Čech-extension of R if S is the closure of R, every point of $S-R$ is a closed set and every function continuous and bounded on R can be extended continuously to S. A Wallman-extension is a $(\omega, \mathfrak{F})$-extension, where $\mathfrak{F}$ is the family of closed sets of R.

In this as in the two preceding papers by the same author, proofs are merely sketched or omitted entirely.
W. L. Ayres (Lafayette, Ind.).

Milgram, A. N. Extensions of coverings from subspaces to spaces. Rep. Math. Colloquium (2) **4**, 16–21 (1943). [MF 8959]

By proving that a separable metric space S is connected between some system of $n+1$ closed subsets if and only if the dimension of S is at least n, the author establishes a relation between the notions of dimension and higher order connectedness, the latter being an extension (due to Menger) of the concept of a space connected between two of its closed subsets. The approach to the theorem is by way of a result of Hurewicz concerning a mapping of a closed subset X of S into the boundary of an n-dimensional simplex not being extensible. Lemmas expressing extensibility of mappings in terms of that of coverings furnish the necessary links. *L. M. Blumenthal* (Columbia, Mo.).

Pepper, Paul M. and Topel, Bernard J. Imbedding theorems under weakened hypotheses. I. Rep. Math. Colloquium (2) **4**, 31–55 (1943). [MF 8961]

This paper is concerned with the congruent imbedding of semimetric spaces S in Euclidean and spherical spaces. Calling a set non-E_k if it is not congruently contained in the Euclidean space E_k of k dimensions, the authors show that each of the following conditions is sufficient in order that S be imbeddable in E_n: (a) S contains at least $n+k+4$ distinct points and at most k non-E_n $(n+2)$-tuples; (b) S contains at least $n+k+3$ distinct points and at most $k(k+3)/2$ non-E_n $(n+3)$-tuples; (c) S is infinite and contains only a finite number of non-E_n $(n+2)$-tuples and $(n+3)$-tuples; (d) S is nondenumerable and has only denumerably many non-E_n $(n+2)$-tuples and $(n+3)$-tuples. [Similar imbedding theorems under weakened hypotheses were obtained by the reviewer [Bull. Amer. Math. Soc. **49**, 321–338 (1943); these Rev. **4**, 251].] Analogous theorems are proved for imbedding in the convex sphere. A set of $n+k+3$ points with exactly k nonmapping $(n+2)$-tuples is called an almost-E_n set. The authors study the structure of such sets, giving an exhaustive survey of almost-E_1 quintuples. *L. M. Blumenthal* (Columbia, Mo.).

Hewitt, Edwin. A problem of set-theoretic topology. Duke Math. J. **10**, 309–333 (1943). [MF 8471]

The author studies various topological spaces defined over some fixed set of points E. If R_1 and R_2 are two such spaces and every open set in R_1 is contained in some open set of R_2, then R_2 is called an expansion of R_1 and R_1 a contraction of R_2. Various types of spaces invariant and noninvariant under all expansions are pointed out. By

means of expansions the collection of topological spaces defined on E forms a partially ordered system. The family of all such spaces forms a complete lattice under this partial ordering. Suppose there exists a completely ordered subfamily of this given family. Then, if any member of it is a T_0-, a T_1-, a Hausdorff- or a Urysohn-space, the least upper bound R_0 of the family enjoys the corresponding property. If every member of the subfamily is either regular or completely regular then so also is R_0.

The remainder of the paper considers only spaces which satisfy the separation axiom of Kolmogoroff. The dispersion character $\Delta(R)$ of R is defined as the least cardinal number of any nonvoid open subset of R. Let τ be a fixed cardinal number and P a property of topological spaces. A topological space R with $\Delta(R)\geqq\tau$ enjoying property P and such that every proper expansion of R either fails to enjoy P or has dispersion character less than τ is said to be τ-maximal with respect to the property P. A space R is τ-maximal if $\Delta(R)\geqq\tau$ and every proper expansion of R has dispersion character less than τ. The space is maximal if it is $\aleph_0$-maximal. Let τ be a cardinal number either infinite or equal to unity and R a topological space with $\Delta(R)\geqq\tau$. Then (a) if R is a T_0-space it has a τ-maximal expansion which is necessarily a T_1-space; (b) if R is a Hausdorff space it has a τ-maximal expansion which is necessarily a Urysohn space (that is, every pair of distinct points of the space is contained in a pair of open sets with disjoint closures); (c) there exist, for every such τ, τ-maximal T_1-spaces and Urysohn spaces; (d) if R is regular (completely regular) then R has an expansion which is τ-maximal with respect to the same property; (e) there exist, for any infinite cardinal number τ, spaces which are τ-maximal with respect to the properties of being regular and completely regular.

A space R devoid of isolated points is resolvable provided it possesses two complementary subsets each dense in R. A space R dense in itself which cannot be resolved into two complementary dense subsets is said to be irresolvable. If R is dense in itself and if no subset of R is resolvable in its relative topology then R is said to be an SI-space or to be SI. If R is dense in itself and if every dense subset of R is open then R is said to be an MI-space or to be MI. It is shown that every maximal space is an MI-space; every MI-space is an SI-space; every SI-space is irresolvable. Every T_0-space dense in itself has a T_1 expansion which is MI. Every Hausdorff space dense in itself has a Urysohn expansion which is MI. Every maximal completely regular space is SI. Every dense subset of an SI-space has a dense interior. Every topological space R devoid of isolated points is the union of two disjoint sets R_1 and R_2, R_1 being a closed set resolvable in its relative topology and R_2 an open set SI in its relative topology. The space R is resolvable if and only if R_2 is void and SI if and only if R_1 is void. Every metric space devoid of isolated points is resolvable. If R is any infinite T_1-space having a character not exceeding its dispersion character, then R is resolvable. Every locally bicompact Hausdorff space devoid of isolated points is resolvable. *D. W. Hall* (College Park, Md.).

v. Alexits, Georg. Über verstreute Mengen. Math. Ann. 18, 379–384 (1942). [MF 8936]

Let S denote a space which is the sum of a countable set of hereditarily locally connected continua, and consider the following classes of sets: (1) the totally disconnected, (2) those having only degenerate quasi-components and (3) the zero-dimensional. It is shown that in S these classes are identical, thereby generalizing certain theorems of G. T. Whyburn [Monatsh. Math. Phys. **38**, 85–88 (1931); Amer. J. Math. **53**, 374–384 (1931)]. Also previous theorems of the author [C. R. Soc. Sci. Varsovie **31**, 104–107 (1938)] for regular curves are generalized; for example, every positive dimensional absolute G_δ or F_σ in S contains an arc. Concerning S itself, it is shown that, in case S is a continuum, then the set of points which fail to have neighborhoods whose closures are hereditarily locally connected continua is nowhere dense. *R. L. Wilder* (Ann Arbor, Mich.).

Kincaid, W. M. On non-cut sets of locally connected continua. Bull. Amer. Math. Soc. **49**, 399–406 (1943). [MF 8387]

Given a locally connected continuum S containing a closed set P such that $S-P$ is connected. Under these hypotheses P can be enclosed in an open set R, the sum of a finite number of regions, whose complement is a locally connected continuum. In addition, if there exists a family of sets F no element of which separates $S-P$, then there exist two open sets R and R' (with $R\supset R'\supset P$) of the above type having the property that no element of F contained in $S-R$ separates $S-R'$. When the elements of F are single points, it is possible to choose $R'=R$; but in general this is not possible. *H. M. Gehman* (Buffalo, N. Y.).

Ettlinger, Martin G. On irreducible continuous curves. Bull. Amer. Math. Soc. **49**, 569–574 (1943). [MF 8858]

Results are obtained for continuous curves in space satisfying axioms 0–1 of R. L. Moore [Foundations of Point Set Theory, Amer. Math. Soc. Colloquium Publ., v. 13, New York, 1932] which are irreducible about closed subsets. The main theorem states: if space satisfies axioms 0–1, and M is an irreducible continuous curve about a compact and closed subset T of M, M is a compact irreducible continuum about T. The reviewer wishes to point out that equivalent results were obtained, under somewhat more general conditions, by R. L. Wilder [Amer. J. Math. **56**, 547–557 (1934), in particular, theorem 13, p. 555]. It is also shown [theorem 2] that in a connected space satisfying axioms 0–2 every closed and compact point set T is a subset of a compact continuous curve. If, in addition, T contains no continuum of condensation the author shows that T is a subset of a compact hereditary continuous curve, and is also a subset of a compact continuous curve which has no continuum of condensation. The former case [theorem 2] has been treated by Wilder for metric space [Fund. Math. **19**, 45–64 (1932), in particular, corollary 1, p. 50]. In the latter case N. E. Steenrod has shown that when such a set is in E_n it is contained in the sum of two arcs [Fund. Math. **23**, 38–53 (1934), in particular, theorem 6, p. 41]. *V. W. Adkisson* (Fayetteville, Ark.).

George, Erich. Eigentlich offene Kurven in regulären Kurvenscharen. Deutsche Math. **6**, 537–542 (1942). [MF 8602]

The paper is concerned mainly with some special cases of results established by the author in a former paper [Deutsche Math. **4**, 462–476 (1939); these Rev. **1**, 29]. *W. Hurewicz* (Providence, R. I.).

Civin, Paul. Two-to-one mappings of manifolds. Duke Math. J. **10**, 49–57 (1943). [MF 8100]

The paper deals with continuous mappings which are exactly two-to-one, in the sense that every inverse image

consists of exactly two points. It has been shown by O. G. Harrold [Duke Math. J. **5**, 789–793 (1939); these Rev. **1**, 223] that such a mapping cannot be defined over an arc. An analogous result for a closed two-cell was obtained later by J. H. Roberts [Duke Math. J. **6**, 256–262 (1940); these Rev. **1**, 319]. In the present paper the author shows that no continuous two-to-one mapping can be defined over a closed three-cell and that furthermore no closed two-to-one mapping can be defined over Euclidean n-space for $n \leqq 3$ (a continuous mapping is called closed if it carries closed sets into closed sets). These results are based on the following theorem which holds for any n-dimensional manifold M (n arbitrary), with or without boundary. Let T be a closed two-to-one mapping of M. For any $x \epsilon M$ denote by $s(x)$ the point determined by the conditions $s(x) \neq x$, $T(s(x)) = T(x)$. Let t be the mapping of the manifold M on itself defined as follows: $t(x) = s(x)$ if the function s is continuous at x, otherwise $t(x) = x$. Then the mapping t is continuous over M. The demonstrations of the foregoing results make extensive use of the well-known fixed point theorems of M. H. A. Newman and P. A. Smith. *W. Hurewicz.*

Shih, Hsiang-Lin. Mappings of 2-manifolds into a space. Duke Math. J. **10**, 179–207 (1943). [MF 8460]

H. E. Robbins [Trans. Amer. Math. Soc. **49**, 308–324 (1941); these Rev. **3**, 141] has given combinatorial conditions that two mappings of a 2-complex K into a space T are homotopic. Shih considers the case that K is a 2-manifold M, obtaining more detailed results. As in Robbins, the mappings f are first deformed into normal mappings: the vertices go into a fixed point, the 1-cells σ^1 into "standard" paths (chosen in advance) representing elements $f(\sigma^1)$ of the fundamental group G of T; f differs from "standard mappings" in the 2-cells σ^2 only in a small patch near the first vertex, and this difference determines an element $h_f(\sigma^2)$ of the second homotopy group π^2 of T. For orientable M, represented by the fundamental polygon $\Delta = s_1 \cdots s_k$, $s_i = a_i b_i a_i^{-1} b_i^{-1}$, define the subgroup

$$\pi_f^2 = \sum f(s_0 \cdots s_{i-1}) \{[1 - f(a_i b_i)]\pi^2 + [f(a_i) - f(s_i)]\pi^2\}$$

of π^2 (G forming a group ring of operators on π^2); similarly for M not orientable. This is a homotopy invariant. [W. Hurewicz suggests (oral communication) that using finite cocycles with G as operator group in the universal covering spaces would materially simplify the definition of π_f^2.] The homotopy of f_1 and f_2 is equivalent to the existence of an extension over $\Delta \times I$, $I = (0, 1)$, of f_1 over $\Delta \times 0$ and f_2 over $\Delta \times 1$, so that a mapping of $M \times I$ is determined. The mapping of $p_1 \times I$ (p_1 = first vertex of Δ) determines $\alpha \epsilon G$. The standard extension over all 2-cells of $\partial(\Delta \times I)$ (including $\Delta \times 0$ and $\Delta \times 1$) determines an element $h(f_1, f_2, \alpha) \epsilon \pi^2$. Theorem: f_1 and f_2 are homotopic if and only if $\alpha \epsilon G$ exists such that

$$f_2(a_i) = \alpha^{-1} f_1(a_i) \alpha, \quad f_2(b_i) = \alpha^{-1} f_1(b_i) \alpha,$$
$$h(f_1, f_2, \alpha) - h_{f_1} + \alpha h_{f_2} \epsilon \pi_{f_1}^2.$$

Let T be the pseudo-projective plane of order m (with fundamental polygon $aa \cdots a$). Standard mappings are chosen, determining the $h(f_1, f_2, \alpha)$. The classification (which is somewhat complicated) is given for M orientable, and for M nonorientable and m odd; also for m = projective plane P and m even. For example, if $M = T = P$, the homotopy classes fall into two distinct sets, an infinite number of classes lying in each. *H. Whitney* (Cambridge, Mass.).

Schweigert, G. E. Minimal A-sets, infinite orbits, and fixed elements. Bull. Amer. Math. Soc. **49**, 754–758 (1943). [MF 9318]

This paper continues the previous work of the author [Proc. Nat. Acad. Sci. U. S. A. **29**, 52–54 (1943); these Rev. **4**, 172] extending to arbitrary homeomorphisms the analysis of Ayres on periodic transformations [Fund. Math. **33**, 95–103 (1939); these Rev. **1**, 45]. In this paper the minimal invariant A-set containing the orbit of E is analyzed in detail, where E is a cyclic element of a semi-locally-connected continuum S which has an infinite period under a homeomorphism $T(S) = S$. In addition, it is proved that, if T is elementwise periodic except on the end points and p is an end point having an infinite orbit, then the minimal A-set containing the orbit of p has a unique fixed element under T and has the cyclic element structure of a n-adic tree. *W. L. Ayres* (Lafayette, Ind.).

Alexandroff, Paul. On homological situation properties of complexes and closed sets. Trans. Amer. Math. Soc. **54**, 286–339 (1943). [MF 9106]

An English translation of a paper previously published in Russian [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 227–282 (1942); these Rev. **4**, 249]. *S. Eilenberg* (Ann Arbor, Mich.).

Bockstein, M. Universal systems of ∇-homology rings. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 243–245 (1942). [MF 8663]

Given a locally bicompact topological space A and a ring R, the cohomology ring $\nabla(A, R)$ is considered. The rings $\nabla(A, I_m)$, where I_m is the ring of integers reduced mod m where $m = 0$ or m is a power of a prime, are of special importance. If m is a divisor of m' a ring homomorphism $\pi_m^{m'} : I_{m'} \to I_m$ is obtained by reducing mod m each element of $I_{m'}$. This induces a ring homomorphism $\pi_m^{m'} : \nabla(A, I_{m'}) \to \nabla(A, I_m)$. An additive homomorphism $\omega_{m'}^m : I_m \to I_{m'}$ ($m' \neq 0$) is obtained by multiplying each integer by m'/m. This induces an additive homomorphism $\omega_{m'}^m : \nabla(A, I_m) \to \nabla(A, I_{m'})$. The collection of rings $\nabla(A, I_m)$ together with the collection of homomorphisms $\pi_m^{m'}$ and $\omega_{m'}^m$ is called the ∇-spectrum of the space A. The author outlines the proof of the theorem that the ∇-spectrum of A determines the rings $\nabla(A, R)$ for arbitrary coefficient rings R. *S. Eilenberg* (Ann Arbor, Mich.).

Fox, Ralph H. On fibre spaces. I. Bull. Amer. Math. Soc. **49**, 555–557 (1943). [MF 8890]

A definition of fibre spaces is given which includes that of Hurewicz and Steenrod [Proc. Nat. Acad. Sci. U. S. A. **27**, 60–64 (1941); these Rev. **2**, 323] and is topologically invariant. The essential idea is to use, together with the space X and continuous mapping π into B (the base space), neighborhoods U of the diagonal set $\sum(b, b)$ in $B \times B$. A definition of uniform homotopy (relative to U) is given. *H. Whitney* (Cambridge, Mass.).

Coxeter, H. S. M. The map-coloring of unorientable surfaces. Duke Math. J. **10**, 293–304 (1943). [MF 8469]

Let U_K be the number of colors sufficient and necessary for the coloring of maps on an unorientable surface of characteristic K and let $F_K = \frac{1}{2}(7 + (49 - 24K)^{\frac{1}{2}})$. The following facts were previously known: $U_K \leqq [F_K]$ always; $U_K = [F_K]$ for $K = +1, -1, -2, -4$ and -5; $U_0 \neq [F_0]$. The present

paper fills a gap by proving $U_{-3}=[F_{-3}]=9$. The necessity of the nine colors is established by constructing a special map with nine regions each touching all the others. The methods of the author of discovering such special maps also simplify the discussion of some of the known cases, particularly the case $K=-5$. *D. C. Lewis.*

NUMERICAL AND GRAPHICAL METHODS

Lowan, Arnold N., Blanch, G. and Abramowitz, M. Table of $Ji_0(x)=\int_x^\infty(J_0(t)/t)dt$ and related functions. J. Math. Phys. Mass. Inst. Tech. **22**, 51–57 (1943). [MF 8944]

Series and asymptotic expansions for the integral $Ji_0(x)$ are obtained and numerical values tabulated in the range $0\leqq x\leqq 10$ at intervals of 0.1, $10\leqq x\leqq 22$ at unit intervals. *M. C. Gray* (New York, N. Y.).

Smith, V. G. An asymptotic expansion of $Ji_0(x)=\int_x^\infty(J_0(t)/t)dt$. J. Math. Phys. Mass. Inst. Tech. **22**, 58–59 (1943). [MF 8945]

An alternative asymptotic expansion for $Ji_0(x)$ [see the preceding review] of the form

$$Ji_0(x)\sim J_0(x)\sum\nolimits_1(x)-J_1(x)\sum\nolimits_2(x),$$

where $\sum_1$ and $\sum_2$ are simple series in powers of $1/x$. This expansion may be more convenient for computation for those values of x for which tables of J_0 and J_1 are available. *M. C. Gray* (New York, N. Y.).

Golomb, Michael. Zeros and poles of functions defined by Taylor series. Bull. Amer. Math. Soc. **49**, 581–592 (1943). [MF 8859]

The classical Hadamard method of determining the poles and zeros of functions defined by power series is here put into a form suitable for numerical calculations. As an example the sum and product of the zeros of $z-e^z$ are calculated to ten significant figures. The results are based on an identity and an inequality for persymmetric determinants involving successive Taylor coefficients of rational and meromorphic functions. *E. C. Titchmarsh.*

Adams, Douglas P. The quintic "hypernom" for the equation $x^5+Ax^3+Bx^2+Cx+D=0$. A graphical method for finding the roots of polynomial equations through the fifth degree. J. Math. Phys. Mass. Inst. Tech. **22**, 78–92 (1943). [MF 8947]

Humbert, Pierre. Solution graphique de l'équation de Képler. C. R. Acad. Sci. Paris **213**, 343 (1941). [MF 9164]

Using a simple construction based on an ellipse and its major auxiliary circle the author solves Kepler's equation to a fair degree of approximation and at the same time gets approximate values for the radius vector and true anomaly. *W. E. Milne* (Corvallis, Ore.).

Lin, Shih-nge. A method for finding roots of algebraic equations. J. Math. Phys. Mass. Inst. Tech. **22**, 60–77 (1943). [MF 8946]

Let $f(x)=x^m+A_{m-1}x^{m-1}+\cdots+A_0$ be a real polynomial, n a positive integer less than m and $b_0, b_1, \cdots, b_{n-1}$ any real constants. Then we may write

$$\sum_{i=0}^{m}A_ix^i=\left(\sum_{i=0}^{n}b_ix^i\right)\left(\sum_{j=0}^{m-n}a_jx^j\right)+\sum_{k=0}^{n-1}r_kx^k,$$

with $a_{m-n}=b_n=A_m=1$; $a_j=0$ for $j>m-n$; $b_j=0$ for $j>n$;

$$A_k=\sum_{j=0}^{k}b_ja_{k-j},\quad k=n, n+1, \cdots, m; \tag{1}$$

$$r_k=A_k-\sum_{j=0}^{k}a_jb_{k-j},\quad k=0, 1, 2, \cdots, n-1. \tag{2}$$

The author presents a method of successive approximations for finding coefficients b_j such that the corresponding $r_k=0$ and thus such that $f(x)$ becomes factored. Let $b_j^{(0)}=0$, $j=0, 1, \cdots, n-1$,

$$A_k=\sum_{j=0}^{k}a_{k-j}^{(p)}b_j^{(p)},\qquad k=n, n+1, \cdots, m, \tag{3}$$

$$A_k=\sum_{j=0}^{k}a_{k-j}^{(p-1)}b_j^{(p)},\quad k=0, 1, 2, \cdots, n-1, \tag{4}$$

and $a_{m-1}^{(p)}=b_n^{(p)}=1$, $a_j^{(p)}=0$ for $j>m-n$ and $b_j^{(p)}=0$ for $j>n$, for all p. Knowing the $a_j^{(p-1)}$, we can compute the $b_j^{(p)}$ from (4), the corresponding $a_j^{(p)}$ from (3), etc. The author asserts that, if the zeros of $f(z)$ are widely separated in modulus, the process converges rapidly. Besides giving many examples, the author discusses alternate procedures for the cases of fourth and eighth degree equations with equal or nearly equal roots. *M. Marden.*

Hillman, A. P. and Salzer, H. E. Roots of $\sin z=z$. Philos. Mag. (7) **34**, 575 (1943). [MF 9137]

Table giving the first ten nonzero roots of $\sin z=z$ in the first quadrant to six decimal places. Obviously the roots are symmetrically situated in the four quadrants.

Rehbock, F. Zur Konvergenz des Newtonschen Verfahrens für Gleichungssysteme. Z. Angew. Math. Mech. **22**, 261–262 (1942). [MF 9122]

Schulz, Günther. Über die Lösung von Gleichungssystemen durch Iteration. Z. Angew. Math. Mech. **22**, 234–235 (1942). [MF 8920]

The author states a sufficient condition for the solvability of a system of two equations in two unknowns by iterations. Let $x=\varphi(x, y)$, $y=\psi(x, y)$ be this system, (a, b) its solution, (x^0, y^0) a first approximation and $x^\nu=\varphi(x^{\nu-1}, y^{\nu-1})$, $y^\nu=\psi(x^{\nu-1}, y^{\nu-1})$ the iterations. Denote by M_x, M_y, N_x, N_y upper bounds for the partial derivatives $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, $\partial\varphi/\partial x$, $\partial\psi/\partial y$ in a convex domain of the xy-plane containing the solution and all the iterations. If the elements of the matrix

$$\begin{Vmatrix} M_x, & M_y \\ N_x, & N_y \end{Vmatrix}$$

are nonnegative and if the solutions of the characteristic equations of this matrix have all an absolute value less than 1, the iterations will converge toward the solution (a, b). To show this the author refers to a lemma which he proved in a previous paper [Z. Angew. Math. Mech. **12**, 44–59 (1932)]. *E. Lukacs* (Jacksonville, Ill.).

Freeman, G. F. On the iterative solution of linear simultaneous equations. Philos. Mag. (7) **34**, 409–416 (1943). [MF 8735]

The author suggests various improvements in the computational techniques for solving linear equations by iterative methods. It is proposed to form a simplified set of equations in which the coefficients are expressed to one decimal place only and the right hand members are divided by a factor 10^n so as to make the greatest among them lie between 1 and 10. This is "the first iterative framework." It is used to iterate the new unknowns to the unit figures. These are substituted in the original equations and, working to two places, the corresponding remainders are found. These, divided by 10^{n-1}, form the right hand members of the "second iterative framework," which is otherwise the same as the first one. It yields as unit solutions the first decimal figures for the unknowns. The combined results express the unknowns to the first place. They are substituted in the original equations and, working to three places, the new remainders are found. They, in turn, are used in a third framework to yield the second decimal place for the unknowns, etc. *W. Feller.*

Collatz, L. Fehlerabschätzung für das Iterationsverfahren zur Auflösung linearer Gleichungssysteme. Z. Angew. Math. Mech. **22**, 357–361 (1942). [MF 9121]

Given a system of n linear equations in n unknowns

$$a_{ik}x^k = r_i, \tag{1}$$

one may use an iteration process to obtain the solution provided $|a_{ii}|$ sufficiently exceeds $|a_{ij}|$ $(j\neq i)$. One method of putting (1) into a suitable form is to multiply (1) on the left by the transpose matrix of the a_{ik}. This is not always satisfactory as not infrequently the convergence may be made worse instead of better. The author therefore prefers Runge's method of linear combination to secure a form giving rapid convergence. The principle result of the paper is a formula which expresses the error of the $(m+1)$th approximation in terms of the difference between mth and $(m+1)$th approximations. *W. E. Milne.*

Meineke, H. Näherungsformel für die Berechnung von Strecken. Z. Angew. Math. Mech. **20**, 359 (1940). [MF 8672]

A study of best (in the sense of least squares) approximations by certain rational functions to (1) the expression $(a^2+b^2)^{\frac{1}{2}}$, $0\leqq a\leqq b$, and (2) the expression for a circular arc in terms of the altitude and chord of the corresponding segment of the circle. *P. W. Ketchum* (Urbana, Ill.).

Vernotte, Pierre. Sur la représentation d'une fonction expérimentale par une fraction rationnelle. C. R. Acad. Sci. Paris **213**, 433–435 (1941). [MF 9168]

The paper discusses a method of representing an experimentally determined function by means of an analytic expression. The latter is taken as a constant plus the derivative of a rational fraction with numerator of degree n and denominator of degree $n+1$. There are then $2n+3$ constants to be determined. (In practice one usually takes $n=1$ or $n=2$.) These constants are determined by equating the mean ordinate of the experimental function in each of $2n+3$ subintervals (usually equal) to the corresponding mean ordinate of the analytic expression. A procedure is outlined for finding the mean ordinates and for solving the resulting equations. *W. E. Milne* (Corvallis, Ore.).

Bennett, H. F. Computation of polynomial functions by summation of finite differences. J. Opt. Soc. Amer. **33**, 519–526 (1943). [MF 8991]

A quick approximate method is described for use with a calculating machine in computing a table of values of a polynomial function which itself may be an approximation to another function. Formulas are given for polynomials up to the eighth degree whereby the computation may be based either upon the coefficients of the polynomial or upon a few equally spaced values of the function. Formulas are given also for predicting the maximum deviation from the exact values of the polynomial. The method is essentially a computing routine which makes use of an arbitrarily defined array of differences and results in replacing the given curve by a series of overlapping parabolic arcs, each extending through five consecutive values of the argument. *Author's summary.*

Egger, Hans. Praktische Interpolation. Z. Angew. Math. Mech. **22**, 362–364 (1942). [MF 9123]

For interpolation in a table of natural sines and cosines the author proposes a method based essentially on the formulas for sin $(x+\Delta x)$ and cos $(x+\Delta x)$. Both direct and inverse interpolation are treated. *W. E. Milne.*

Knobloch, H. Zur Interpolation von Kurvenscharen. Z. Angew. Math. Mech. **22**, 364–366 (1942). [MF 9124]

Given a few computed curves from a system of plane path curves $x=x(t, \lambda)$, $y=y(t, \lambda)$, in which λ is a parameter (for example, the paths of a projectile with different angles of departure λ), the problem is to obtain intermediate paths by some process of interpolation. The author first cites the methods of Athen, Sauer and Stange, using affine transformations of the x, y-plane to facilitate interpolation, and then proposes other methods based on an analytical solution of a simplified but approximately similar dynamical problem. For example, one might use the solution of the ballistic problem with resistance proportional to velocity, which can be completely solved. By proper adjustment of the parameters the difference between the actual computed solution and the approximate analytical solution can be made small, and interpolation of this small difference can be relatively accurate. The method is illustrated by applications to horizontal bombing and dive bombing. *W. E. Milne.*

Salzer, Herbert E. Coefficients for numerical differentiation with central differences. J. Math. Phys. Mass. Inst. Tech. **22**, 115–135 (1943). [MF 9253]

This table gives the values of the coefficients at $x=0$ (that is, at the central value) for the nth derivative of Stirling's central difference interpolation formula. Coefficients are given for derivatives of all orders as far as the 52nd, and for all differences of orders as far as the 42nd. For derivatives of orders $n=33$ to $n=42$, coefficients are given for differences of orders as far as $n+10$. For higher order derivatives the coefficients are given as far as the 52nd order difference. The exact fractional value of the coefficient is given in all cases as far as coefficients of order 30, while for higher orders some of the coefficients are given in decimals with 18 significant figures. *W. E. Milne.*

Lowan, A. N. and Salzer, Herbert. Table of coefficients in numerical integration formulae. J. Math. Phys. Mass. Inst. Tech. **22**, 49–50 (1943). [MF 8943]

This table gives the first 20 coefficients in Gregory's formula for numerical integration with forward differences

and the first 20 coefficients in Laplace's formula for numerical integration with backward differences.
W. E. Milne (Corvallis, Ore.).

v. Borbély, S. Über die praktische Integration ebener Vektoren. Z. Angew. Math. Mech. **22**, 273–277 (1942). [MF 8912]

A procedure is given for the graphical integration of a complex function of a real variable. The geometrical steps of the method are carried out in the complex plane, so that no use is made of the possibility of carrying out instead two real component integrations. The method is based on the formula

$$\int_0^h r(\phi)e^{i\phi}d\phi \approx (1/6)[r(h)+4r(h/2)+r(0)]\int_0^h e^{i\phi}d\phi + (ih/12)[r(h)-r(0)]\int_0^h e^{i\phi}d\phi,$$

which is correct up to terms of order h^3.
E. Reissner (Cambridge, Mass.).

Collatz, L. Natürliche Schrittweite bei numerischer Integration von Differentialgleichungssystemen. Z. Angew. Math. Mech. **22**, 216–225 (1942). [MF 8987]

In any given step-by-step process of numerical integration of a given differential equation the accuracy of the process depends on the length of the step. If the step used is too large the result is subject to error and the labor is increased because of slow convergence and consequent numerous recalculations. If the step is too small the unnecessarily large number of steps multiplies the labor. Employing the results of previous investigations on the progressive growth of the error, notably those of R. v. Mises [Z. Angew. Math. Mech. **10**, 81–92 (1930)], G. Schulz [ibid. **12**, 44–59 (1932)] and Collatz and Zurmühl [ibid. **22**, 42–55 (1942); these Rev. **4**, 149], the author develops methods for determining the optimum step length in any particular problem. The case of systems of differential equations is given detailed attention. *W. E. Milne* (Corvallis, Ore.).

Collatz, L. und Zurmühl, R. Zur Genauigkeit verschiedener Integrationsverfahren bei gewöhnlichen Differentialgleichungen. Ing.-Arch. **13**, 34–36 (1942). [MF 8597]

In Adams' method for the numerical integration of $y'=f(x,y)$, and of $y''=f(x,y,y')$ still more, various interpolating polynomials and different intervals of integration can be used. The authors give the error terms (first neglected term in the Taylor expansion) for the most commonly used combinations. *W. Feller* (Providence, R. I.).

Kovner, S. S. On the technique of numerical integration of differential equations with partial derivatives. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 20–23 (1942). [MF 8495]

The author is concerned with the finite difference method for solving the heat equations. He describes in a very sketchy way arrangements and short cuts which his experience has shown to be most practical. *W. Feller.*

Lowan, Arnold N. and Laderman, Jack. Table of Fourier coefficients. J. Math. Phys. Mass. Inst. Tech. **22**, 136–147 (1943). [MF 9254]

This table gives to 10 places of decimals the values of the Fourier coefficients

$$\int_0^1 x^k \sin n\pi x dx, \quad \int_0^1 x^k \cos n\pi x dx$$

for all integral values of k from 0 to 10 inclusive and for all integral values of n from 1 to 100 inclusive.
W. E. Milne (Corvallis, Ore.).

Ross, M. A. S. Numerical Fourier analysis to twenty-nine harmonics. Nature **152**, 302–303 (1943). [MF 9094]

Computing schedules are developed for evaluation of Fourier harmonic components. The central feature of the method is the application of Beevers and Lipson [Proc. Phys. Soc. **48**, 772 (1936); Nature **137**, 825 (1936)] strips in computing harmonic coefficients from Bessel's solution. Formulas are given for checking the computations. The time for performing a complete computation and check for twenty-nine harmonics is given as less than two hours.
S. H. Caldwell (Cambridge, Mass.).

Berger, Erich Rud. Harmonische Analyse diskreter Zahlenreihen. Z. Angew. Math. Mech. **22**, 269–272 (1942). [MF 8911]

Let a_n, b_n denote the coefficients in the trigonometric interpolation formula for m equally spaced ordinates. Let a_n^k, b_n^k denote the Fourier coefficients formed for the curve C_k obtained by joining the successive ordinates by arcs of polynomial curves of kth degree so as to make the derivative of $(k-1)$th order continuous. Simple relations connect a_n and a_n^k, b_n and b_n^k. Hence a_n and b_n can be found by using a harmonic analyzer on the curve C_k. By a reversal of the process one may use the harmonic analyzer to obtain equally spaced ordinates for a Fourier series with known coefficients. *W. E. Milne* (Corvallis, Ore.).

Kelley, Truman L. The evidence for periodicity in short time series. J. Amer. Statist. Assoc. **38**, 319–326 (1943). [MF 9047]

After a polynomial trend has been subtracted off, the residuals are arranged in periods [Buys-Ballot scheme; cf. Karl Stumpff, Grundlagen und Methoden der Periodenforschung, Springer, Berlin, 1937]. Analyzing variance in this table, the author plots the square ϵ^2 of the correlation ratio, or the likelihood integral P, against the period T to form his periodogram. The method is independent of wave form and is less affected by autocorrelation than most methods (and in the opposite direction), but is subject to the disadvantage that similarities in the data in neighboring but not identical phases (modulo T) are not permitted to contribute positively to the ordinate of the periodogram, but, in fact, detract from it somewhat. *A. Blake.*

Frolow, Vladimir. Utilisation du coefficient de corrélation dans l'analyse harmonique. C. R. Acad. Sci. Paris **213**, 56–57 (1941). [MF 9153]

Straiton, Archie W. and Terhune, George K. Harmonic analysis by photographic method. J. Appl. Phys. **14**, 535–536 (1943).

This paper describes a method for determining Fourier series coefficients from area measurements on photographs of the function to be analyzed with the graph sheet formed into a series of semi-cylinders. *R. L. Dietzold.*

Sreedharan Pillai, K. C. Trend analyser. Proc. Indian Acad. Sci., Sect. A. **17**, 187–194 (1943). [MF 9264]

The author describes a mechanical device for the trend analysis of a time series. The given points of the time series are represented by movable rings sliding on vertical parallel

rods representing equally spaced ordinates. By a system of springs each of these rings resists displacement vertically by a force proportional to the displacement. The rods are first adjusted so that in the equilibrium position the series of rings represents the graph of the original data. Then a flexible wire is passed through the rings and tension applied, displacing the rings toward a smooth curve. This curve is taken as the desired approximating curve. *W. E. Milne.*

MATHEMATICAL PHYSICS

Szymański, Piotr. Essai sur la théorie mathématique des sensations de couleur. Mathematica, Timişoara **19**, 69–96 (1943). [MF 9425]

Synge, J. L. On Herzberger's direct method in geometrical optics. Quart. Appl. Math. **1**, 268–272 (1943). [MF 9367]

Herzberger's direct method [Trans. Amer. Math. Soc. **53**, 218–229 (1943); Quart. Appl. Math. **1**, 69–77 (1943); these Rev. **4**, 204] for analytical ray tracing through rotationally symmetric systems is compared and combined with Hamilton's method. A formulation is given of the problem of determining the Herzberger transformation when Hamilton's angle-characteristic is known. The case of a single surface (refracting or reflecting) is considered and a new relation is found connecting the coefficients. Finally, Herzberger's geometrical approach to the problem of the sphere is replaced by an analytical method. *P. Boeder.*

Irons, Eric J. Notes on refraction at a plane interface and by a (triangular) prism. Philos. Mag. (7) **34**, 608–614 (1943). [MF 9333]

The well-known method of graphical ray tracing by constructing two concentric circles of radii n_1 and n_2 (the refractive indices of the materials separated by a plane interface) is used (1) to find n_2, given n_1, and some experimentally determined simultaneous values of i_1, and i_2, (2) to investigate the deviation produced by a prism, (3) to show that the angle of minimum deviation for a prism occurs when the ray traverses the prism symmetrically. *P. Boeder.*

Korff, Günther. Das Prinzip der Totalundeutlichkeit bei der trigonometrischen Durchrechnung. Z. Instrumentenkunde **63**, 1–8 (1943). [MF 8807]

In a previous publication [Zum Gaussschen Prinzip der Totalundeutlichkeit, same Z. **61**, 208–212 (1941)] the author has proven that there does not exist a generally valid correction of an optical system which is entirely independent of all special properties of the system. The principal equation of the paper quoted could, however, not be applied to the method of trigonometric ray tracing. In the present paper another formulation is given which permits the application of the "Prinzip der Totalundeutlichkeit" to the trigonometric method. *P. Boeder* (Southbridge, Mass.).

Kuznetzov, E. S. On approximate equations of transfer of radiation in a scattering and absorbing medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 209–214 (1942). [MF 8662]

The equations of Schwarzschild and Milne [Handbuch der Astrophysik, part 3, vol. 1, Springer, Berlin, 1930, p. 165] for propagation of radiant energy in a scattering and absorbing medium are applied to the case of a radiation field which is a function of one coordinate. The equations are transformed by several substitutions into a system of two differential equations and one integral equation in three unknown functions which are the two fluxes (integrals of intensities over two complementary hemispheres of directions) and the emission coefficient. Under certain further assumptions the three equations become linear. It is shown how these assumptions can be made to correspond systematically to the special cases previously studied, such as those of spherical scattering and Rayleigh scattering, and it is shown how sensitive are the assumptions made to changes in the boundary conditions. *W. Kaplan.*

Kuznetzov, E. S. On the problem of light propagation in the sea. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 10–13 (1943). [MF 8693]

The author indicates the physical significance of the system

$$(d/dt)F^i=\sum_{j=1}^{2} A_{ij}F^j, \qquad i=1, 2,$$

in problems of light propagation in the sea. For the case A_{ij} constant it is shown that the coefficient of reflection F^2/F^1 is independent of depth for an infinitely deep sea.

D. G. Bourgin (Urbana, Ill.).

Ambarzumian, V. A. Diffuse reflection of light by a foggy medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 229–232 (1943). [MF 9246]

Krat, V. Some problems of light scattering in the earth's atmosphere. Astr. J. Soviet Union **19**, 18–29 (1942). (Russian. English summary) [MF 9108]

The author deduces necessary formulas for the brightness of the diurnal sky in different directions. These formulas are applied to the case of an atmospheric model consisting of two uniform layers. The light scattering of higher orders is evaluated qualitatively. A simple method is proposed to account for the reflexion of light from terrestrial objects, particularly from snow, etc. In addition formulas have been developed for calculating the amount of the aerial haze. *Author's summary.*

Newton, Robert R. Distribution of light intensity within a scattering medium. J. Appl. Phys. **14**, 481–486 (1943).

The paper presents the development of an integrodifferential equation which defines the distribution of light intensity in a layer of a scattering medium as a function of position and direction. By means of this equation the variation of intensity can be calculated by methods of successive approximations, provided the scattering properties of an infinitesimal volume of the layer are known. Concerning these the only assumption made is that the amount of energy scattered is proportional to the thickness of the infinitesimal volume and depends on the direction of scattering. Several special cases are discussed, an approximate criterion for the required lateral extent of the sample is given and further uses of the function are indicated.

P. Boeder (Southbridge, Mass.).

Silberstein, Ludwik. Simplified formulae for scattered and rescattered sunlight. J. Opt. Soc. Amer. **33**, 526–532 (1943). [MF 8992]

The formulae of A. Hammond and S. Chapman [Philos. Mag. (7) **28**, 99–110 (1939)] are simplified by introducing

certain reductions and discarding unimportant terms. The formula for the reception of the primary scattered sunlight is obtained for the case where the receiving element is placed at the ground level. The formula for the reception of the secondary scattered light is obtained for the case of the sun in zenith. The relative importance of the primary and secondary scatterings is considered. The tertiary scattering is neglected altogether because it is considered less important than the error introduced by the neglected sphericity of the earth and the refraction of light in its atmosphere. *P. Boeder* (Southbridge, Mass.).

Schrödinger, Erwin. A new exact solution in non-linear optics. (Two-wave-system). Proc. Roy. Irish Acad. Sect. A. **49**, 59–66 (1943). [MF 8997]

The interaction of two plane circularly polarized waves in nonlinear electrodynamics is derived. The solution is exact, that is, obtained in closed form, and is simpler than the first perturbation-approximation given in an earlier paper [Proc. Roy. Irish Acad. Sect. A. **47**, 77–117 (1942); these Rev. **4**, 31]; except for the static centrosymmetric solution representing the "Born-electron," it is so far unique in this field. Application of a suitable Lorentz transformation will produce a frame in which the two wave trains appear antiparallel, which case is assumed at the outset. The solution is obtained in straightforward manner by showing that there exist superpositions of two such wave trains with polarization vectors $\mathfrak{a}$ and $\mathfrak{a}^*$ and wave vectors $\mathfrak{k}$ and (real const.)$\cdot\mathfrak{k}$, respectively, which will satisfy the conditions of conjugateness of nonlinear electrodynamics based on the Lagrangian $L=(\mathfrak{F}^2-\mathfrak{G}^2)/(\mathfrak{F}\cdot\mathfrak{G})$ [see reference] if only the respective amplitudes are chosen appropriately, their moduli being obtained explicitly as functions of the two phase velocities A_i; $A^2=1-E^2-H^2$; the success is due entirely to the fact that the products of the magnetic six-vector components of each wave with their scalar products reproduce the other wave. The result is particularly interesting in so far as it shows that "abnormal waves," that is, those with $A<0$ (antiparallelism of field and displacement vector, etc.) (whose physical interpretation is still a question mark), must be included, that is, do "occur" under full relativistic invariance; indeed, it is shown that "given a couple of antiparallel normal waves, however weak, you can always indicate a frame of reference in which one of them has its direction reversed, dragged along, as it were, by the other one and exhibiting the features of abnormality. In one particular frame it becomes petrified, static, as it were. Moreover you are, in every case, free to choose which of the two you want to subject to such extremity."
H. G. Baerwald (Cleveland, Ohio.).

La Paz, Lincoln and Miller, Geoffrey A. Optimum current distributions on vertical antennas. Proc. I.R.E. **31**, 214–232 (1943). [MF 8818]

The physical problem is the determination of the current distribution on a vertical antenna to give maximum radiated power on the horizon. Mathematically this is the problem of minimizing

$$\text{(a)}\qquad P(f)=\int_0^{\pi/2}\sin^2\theta\left(\int_0^L f(x)\cos(x\cos\theta)dx\right)^2 d\theta$$

subject to (b) $\int_0^L f(x)dx$ a constant. It is shown that a solution satisfies

$$\text{(c)}\qquad \int_0^L f(x)\left[\frac{1}{z}\frac{d}{dz}\left(\frac{\sin z}{z}\right)\right]_{z=x-y}dx=C$$

for $0\leqq y\leqq L$. The authors show $P(f)$ has an inf but do not settle whether a minimizing function exists. Practical methods for determining a minimizing sequence are indicated. *D. G. Bourgin* (Urbana, Ill.).

Pincherle, L. Reflexion and transmission by absorbing dielectrics of electromagnetic waves in hollow tubes. Philos. Mag. (7) **34**, 521–532 (1943). [MF 9132]

Formulas are developed for the reflected and transmitted field intensities in a rectangular wave guide with one section filled with an absorbing dielectric. Neglecting losses in the walls, it is shown that the formulas for amplitudes and phase changes on reflection are the same as can be derived by applying the well-known Fresnel formulas of refraction and reflection to the component plane waves into which the field pattern can easily be analyzed. The formulas are then applied to dielectric sheets of finite thickness extending across the wave guide. *E. Weber* (Brooklyn, N. Y.).

Flint, H. T. and Pincherle, L. The impedance of hollow wave guides. Proc. Phys. Soc. **55**, 329–338 (1943). [MF 9024]

Using the impedance definitions given by S. Schelkunoff [Proc. I.R.E. **25**, 1457–1492 (1937)], the authors apply them to the wave guides with dielectric plungers of finite thickness. They suggest a measurement of the open end impedance by means of such dielectric plungers.
E. Weber (Brooklyn, N. Y.).

Cauer, W. Bemerkung über eine Extremalaufgabe von E. Zolotareff. Z. Angew. Math. Mech. **20**, 358 (1940). [MF 8671]

The author gives a number of references to applications of extremal problems in electrical engineering, especially the application to electrical filter problems of E. Zolotareff's solution [Collected Works (in Russian), vol. 1, 1931, p. 372; vol. 2, 1932, pp. 1–60] of an extremal problem of Tschebyscheff. The author also states that a derivation (not a mere verification) of Zolotareff's solution will appear in his forthcoming book. *M. A. Basoco* (Lincoln, Neb.).

Kessenich, B. N. A theorem on the energy stored in a reactive two-pole. Acad. Sci. USSR. J. Phys. **7**, 37–41 (1943). [MF 8886]

A proof is given of the theorem that, upon switching off a sinusoidal electromotive force which maintained a stationary state in a network consisting of a resistance in series with a pure reactive network, the energy dissipated in the resistance is determined by the values of the reactance and of its derivative at the operating frequency, and is thus independent of the functional form of the reactance. This theorem had been presented by the author, with an incomplete proof, in an earlier paper [same J. **4**, 123–142 (1941)].
R. M. Foster (New York, N. Y.).

Grünberg, G. A. General theory of the focusing action of electrostatic and magnetostatic fields. I. Two-dimensional fields. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 172–178 (1942). [MF 8659]

In this paper the author considers the general properties of trajectories of an electron moving in a two-dimensional electrostatic and magnetostatic field. Utilizing the equation of normal acceleration and the equation of energy, he expresses the normal derivative $\partial\varphi/\partial n$ of the potential along a trajectory in terms of the potential φ and the curvature

ρ of the trajectory. From this relation it follows that, if two trajectories intersecting in two points A, C are given, then within the area enclosed by them the potential φ is determined from the relation between φ and $\partial\varphi/\partial n$ over the boundary. The remaining trajection will then be determined, and in general the paths through A will not pass through C. If one trajectory and "a neighboring" one are given, then the potential is also determined. A differential equation is obtained between the potential φ along the trajectory and the distance ρ to the neighboring trajectories. From this it is possible to determine either one of the functions ρ, φ in terms of the other one. This equation, in the reviewer's opinion, is analogous to the first order of focusing commonly derived for the neighborhood of a straight line trajectory which it generalizes by replacing the straight line by an arbitrary curve. *H. Poritsky.*

Grünberg, G. A. General theory of the focusing action of electrostatic and magnetic fields. II. Three-dimensional electrostatic fields. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 261–268 (1942). [MF 8666]

This paper continues the investigation of the one reviewed above. Three-dimensional electrostatic fields are investigated; in particular, the properties which a family of curves must satisfy to serve as such trajectories for a proper field are studied. By setting up a moving system of coordinate axes whose directions are those of the tangents, principal normal and binormal or a particular trajectory and denoted by p_2, p_3, the displacement components of a nearby trajectory in the latter two directions, the author obtains two simultaneous differential equations for p_2, p_3 and the potential φ and its first and second derivatives $d\varphi/ds$, $d^2\varphi/ds^2$ along the trajectory. These equations limit the arbitrariness and the choice of a family of trajectories. The results are examined for the case of a field of rotational symmetry. *H. Poritsky* (Schenectady, N. Y.).

Grünberg, G. A. General theory of the focusing action of electrostatic and magnetic fields. III. Three-dimensional (twisted) trajectories in the presence of both an electrostatic and a magnetic field. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 78–81 (1943). [MF 8690]

[Cf. the two preceding reviews.] In this paper the author discusses electron paths in simultaneous electrostatic and magnetostatic fields by a method similar to that of the preceding paper. Again two simultaneous differential relations are obtained involving the distances p_2, p_3 of a trajectory along the principal normal and binormal of a particular trajectory, and the potentials, both electrostatic and magnetic, and their derivatives. *H. Poritsky.*

Eliezer, C. Jayaratnam. The hydrogen atom and the classical theory of radiation. Proc. Cambridge Philos. Soc. **39**, 173–180 (1943). [MF 9199]

Dirac's classical theory of the point electron with radiation damping is applied to the problem of the electron moving in a straight line towards a fixed proton and the rectilinear motion of two electrons. The solution for the two and three dimensional motions of an electron about a fixed proton are also discussed. There appears to be no solution which would permit a collision between two particles except in the case of the rectilinear collision of two electrons of like charges. Thus the results are in all cases opposite to what we should expect from elementary physical considerations. *S. Kusaka.*

Brillouin, Léon. Theory of the magnetron. I. Phys. Rev. (2) **60**, 385–396 (1941). [MF 8047]

This paper considers the simplest type of magnetron, consisting of a filament of radius a surrounded by a cylindrical anode of radius b, with a magnetic field applied parallel to the axis of the filament. The static case, in which the potential V is solely a function of the distance r from the axis, is first discussed and it is shown that there is a maximum value of the anode potential for which the electrons emitted from the filament have no radial velocity but there is a space charge throughout the medium between filament and anode. At potentials above the critical potential there is a current I in the magnetron and the differential equation for the potential can be solved only approximately. A characteristic length L is found, defining a distance beyond which the electrons describe spirals round the filament. Potential curves show that when $a<L<b$ the electrons are accelerated from a to L and retarded from L to b.

The author next considers the quasi-static case of small perturbations of the above static solution. When the anodic current is very small the electrons will have a proper frequency of oscillation nearly equal to $\sqrt{2}\omega_H$, where ω_H is the Larmor angular velocity. Assuming sinusoidal current distribution, a formula for the internal impedance of the magnetron is obtained and conditions for a negative resistance component of this impedance are discussed. *M. C. Gray.*

Brillouin, Léon. Theory of the magnetron. II. Oscillations in a split-anode magnetron. Phys. Rev. (2) **62**, 166–177 (1942). [MF 8048]

This paper discusses mainly the static theory for a magnetron in which the anode consists of $2n$ equispaced sections of a cylinder of radius b. The anodes have an average constant potential V_e plus an additional alternating potential which is alternately positive and negative on successive anode sections and is small compared with V_e. It is assumed that the resulting electric field may be expressed as the gradient of a scalar potential. Equations for the components of this potential are first found for the static case and the theory of small oscillations in the split-anode magnetron then developed by a method similar to that used in the paper reviewed above. Elementary solutions for the alternating potential function are found to be of the form $V_a=Kr^x$, where x is a root of a quadratic equation whose coefficients depend only on the frequency-ratio $y=\omega/\omega_H$ and on the number n. A discussion of the roots of the quadratic shows that there are regions in the $y-n$ plane where large oscillations may be obtained while in other regions no oscillations are possible. Formulas for the current densities on the anodes and for the internal impedance of the magnetron are also obtained. *M. C. Gray.*

Brillouin, Léon. Theory of the magnetron. III. Phys. Rev. (2) **63**, 127–136 (1943). [MF 8049]

This paper returns to the magnetron of paper I above and gives a more detailed discussion of the possible oscillations when the radius of the filament is not neglected. It is shown that the magnetron can sustain oscillations on the frequency

$$\bar{\omega}=\omega_H(2+2a^4/b^4)^{\frac{1}{2}}.$$

When the direct current I_e is so small that the characteristic length L is small compared with a, the alternating potential is expressed as a complex integral, and most of the paper discusses approximate values of this integral. In this region the internal resistance of the magnetron may be negative

in certain narrow frequency bands just below the frequency $\bar{\omega}$. When the magnetron is operating below the critical potential ($I_s=0$), it behaves like an inductance plus a positive resistance in the frequency range $\bar{\omega}\leqq\omega\leqq 2\omega_H$; outside this range it behaves like an inductance. The author concludes this paper with a short discussion of the plane magnetron. *M. C. Gray* (New York, N. Y.).

Pauli, W. and Kusaka, S. On the theory of a mixed pseudoscalar and a vector meson field. Phys. Rev. (2) **63**, 400–416 (1943). [MF 8483]

The theory of nuclear forces based on a mixture of pseudoscalar and vector meson field is investigated in the strong coupling case. The paper consists largely of detailed algebraic computations. The isobar separation is shown to have $\frac{1}{3}$ of its value in the pseudoscalar theory. The lowest state for the deuteron is found to be the triplet and other states are classified. A suitable choice of coupling constant and of the masses of the mesons gives the observed values of the binding energy and the quadrupole moment of the deuteron. However, the magnetic moment obtained proves to be only a few per cent of the observed value. It is concluded that the strong coupling theory should be abandoned in favor of a weak coupling theory, with the singularities due to the point sources eliminated by a subtraction formalism. *G. C. McVittie* (London).

Iwanenko, D. and Sokolov, A. On the dipolness of mesons and the difficulties of the Proca theory. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **12**, 473–478 (1942). (Russian) [MF 8409]

It follows from the vector theory of mesons that a particle obeying Proca's equations behaves as if endowed with a dipole moment with respect to both the electromagnetic and the nuclear forces. The authors show in detail that the meson's dipole moment is the source of many difficulties of the vector theory of mesons (high singularities of the potential fields and large cross sections for various processes). The hypothesis of W. Heitler and S. T. Ma [Proc. Roy. Soc. London. Ser. A. **176**, 368 (1940)] concerning excited states of the "nucleon" with different values of spin and charge removes only the difficulties connected with the nuclear forces. In a concluding remark the authors put forward the idea that a similar hypothesis for the meson might help to overcome the remaining difficulties which are directly connected with the electromagnetic field. *V. Bargmann.*

Langevin, Paul. Sur les chocs entre neutrons rapides et noyaux de masses quelconques. Ann. Physique (11) **17**, 303–317 (1942). [MF 9228]

Morgans, W. R. On the capture of a slow-moving directed electron in a Coulomb field of force. Philos. Mag. (7) **34**, 537–549 (1943). [MF 9134]

Parabolic coordinates are used. The solution of the Schrödinger wave equation for electrons roaming in the direction of the z-axis, due to Temple and Epstein, is recalled and the components of the corresponding Heisenberg matrices are obtained. From these the probability of spontaneous transition and of the absorption coefficients is calculated and the actual intensity associated with transitions between these states is found. The limiting case for the intensity for large quantum numbers e and for slow moving electrons is obtained by means of the saddle point method and the result expressed in closed form.

H. Poritsky (Schenectady, N. Y.).

Ginsburg, V. On the theory of a particle with a spin $\frac{3}{2}$. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **12**, 425–442 (1942). (Russian) [MF 8405]

The wave equations for a particle with spin $\frac{3}{2}$ have been established in spinor form by P. A. M. Dirac [Proc. Roy. Soc. London, Ser. A. **155**, 447 (1936)] and M. Fierz [Helvetica Phys. Acta **12**, 3–37 (1939)]. The present paper is devoted to a detailed discussion of the wave equations. Following a method suggested by I. E. Tamm and worked out by A. S. Davydov [not yet published] the author first rewrites the equations in a form similar to that of Dirac's equation of the electron. The two component spinors are replaced by four four-component spinors A^l ($l=1, \cdots, 4$) which define a mixed spinor-vector quantity, l being an ordinary vector index. Then the wave equations read

$$\gamma^r\partial A_l/\partial x^r+\kappa A_l=0, \quad \kappa=mc/\hbar; \quad l=1, \cdots, 4, \tag{1}$$

$$\gamma^r A_r=0, \tag{2}$$

where γ^r are the well-known Dirac matrices. In order to derive the wave equations for a particle with spin higher than 1 from a variation principle and describe the interaction with an external electromagnetic field it is necessary to introduce additional quantities into the theory, as has been shown by M. Fierz and W. Pauli [Proc. Roy. Soc. London, Ser. A. **173**, 211–232 (1939); these Rev. **1**, 190]. The Lagrangian corresponding to (1) and (2) is then given by

$$L_{3/2}=A_l^+\gamma^r\frac{\partial A^l}{\partial x^r}+\kappa A_l^+A^l$$
$$-i\left(A^l+\frac{\partial C}{\partial x^l}-\frac{\partial C^+}{\partial x^l}A^l-\frac{2}{3}C^+\gamma^r\frac{\partial C}{\partial x^r}\right)+3\kappa C^+C,$$

where $A_m^+=A_m^*\gamma^4$ and C is an additional four-component spinor. For the variation of $L_{3/2}$ the quantities A_m, A_m^+, C and C^+ are treated as independent, the A_m being subject to the subsidiary condition $\gamma^m A_m=0$. If an electromagnetic field is present, the derivatives $\partial/\partial x^l$ must be replaced by the operators

$$\Pi_l=\frac{\partial}{\partial x^l}-\frac{il}{\hbar l}\Phi_l$$

(where Φ_l are the components of the vector potential). The author derives the expressions for the four vector current and the energy-momentum tensor and discusses the corresponding conservation laws. The equations for a free particle are solved explicitly. In the last two sections the author deals with the magnetic properties of the particle and its interaction with the electromagnetic field (by means of a perturbation theory). *V. Bargmann.*

Ginsburg, V. L. Relativistic wave equations for particles with variable spin. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 166–171 (1942). [MF 8658]

Using the results of a previous paper [cf. the preceding review] the author establishes relativistic wave equations with variable spin. [According to the hypothesis of W. Heitler and S. T. Ma [Proc. Roy. Soc. London, Ser. A. **176**, 368–397 (1940)] the proton is a particle of this type.] He studies the two cases ($\frac{1}{2}$, $\frac{3}{2}$) and (1, 2). (By (i, j) he denotes a particle whose spin might assume the values i and j.) (a) The particle ($\frac{1}{2}$, $\frac{3}{2}$) is described by the four-component spinor ψ, and the quantities A_l and C [cf. the preceding review]. For the free particle the wave equations are derived from the Lagrangian

$$L_{(1/2,\,3/2)}=L_{1/2}+L_{3/2}+\alpha(C^+\psi+\psi^+C)$$

with

$$L_{1/2}=\psi^+\gamma^l\partial\psi/\partial x^l+k\psi^+\psi.$$

In the case of an external electromagnetic field $\partial/\partial x^l$ is replaced by Π_l. It is necessary to restrict the constant α by the inequality $\alpha^2<6\kappa^2$ in order to obtain a positive definite total charge. (b) The particle (1, 2) is described by the four vector U_i, the symmetric tensor A_{ik} with vanishing trace and the scalar C. Its Lagrangian is given by

$$L_{(1,2)}=L_1+L_2$$
$$+\alpha\left(A^*_{ik}\frac{\partial U_i}{\partial x^k}+A_{ik}\frac{\partial U_i^*}{\partial x^k}\right)+\tfrac{3}{4}\alpha\left(U_i^*\frac{\partial C}{\partial x^i}+U_i\frac{\partial C^*}{\partial x^i}\right);$$

L_1 is the Lagrangian for a particle with spin 1 (leading to Proca's equations for U_i) and L_2 the Lagrangian for a particle with spin 2, which has been derived by M. Fierz and W. Pauli [Proc. Roy. Soc. London, Ser. A. 173, 211–232 (1939); these Rev. 1, 190]. *V. Bargmann.*

Ginsburg, V. and Nemirovskij, P. Wave equation for a particle with a spin ½ and with two values of the rest mass. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 12, 443–448 (1942). (Russian) [MF 8406]

Starting from an appropriate Lagrangian, the authors derive wave equations for a particle with spin ½ and two different values of the rest mass (possibly proton-neutron). The wave function has 8 components and can be represented by two four-component spinors. In the authors' opinion it is doubtful whether these equations can be utilized in a theory of the excited states of heavy particles. [Reviewer's note. The wave equations discussed in this paper coincide with those derived by A. Einstein and W. Mayer in their theory of semivectors [Nederl. Akad. Wetensch., Proc. 36, 496–516, 615–619 (1933)]. In particular, it has been shown by Einstein and Mayer that these equations can be decomposed into two independent systems of Dirac equations [cf. loc. cit., p. 619, eq. (16), (17)]. The spinor form of these equations has also been studied by V. Bargmann [Helvetica Phys. Acta 7, 57–82 (1934), in particular, §8].] *V. Bargmann* (Princeton, N. J.).

Ginsburg, V. On the pseudoscalar theory of mesons. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 12, 466–472 (1942). (Russian) [MF 8408]

From the pseudoscalar theory of the meson the author derives cross sections for various processes involving mesons. In particular, he considers (a) the scattering of mesons by a proton, (b) the creation of meson pairs by photons in a nuclear field. In contradiction with experiment, the cross sections of these processes increase rapidly with the energy E of the incident particle. If $E\gg\mu c^2$ (where μ denotes the meson mass) one finds for (a) a cross section proportional to $(E/\mu c^2)^2$, for (b) a cross section proportional to $(E/\mu c^2)^2\log(E/\mu c^2)$, and for the "Bremsstrahlung" of mesons a cross section of the same order of magnitude. As W. Heitler and S. T. Ma [Proc. Roy. Soc. London, Ser. A. 176, 368–397 (1940)] have shown in the case of the vector theory of mesons, one obtains a finite value for the cross section (a) by assuming for the heavy particles the existence of excited states with different values of spin and charge. The question whether this assumption leads also to a finite cross section for the process (b) will be discussed by the author in a future investigation in which he intends to study in detail a relativistic theory of particles with variable spin. *V. Bargmann* (Princeton, N. J.).

Biswas, B. N. Application of photon statistics to the specific heat of a monatomic solid. Indian J. Phys. 26, 1–5 (1943). [MF 8957]

Wannier, Gregory H. Some remarks on the statistics of binary systems. Proc. Roy. Soc. London. Ser. A. 181, 409–411 (1943). [MF 8851]

Fuchs, Klaus. On the statistics of binary systems. Proc. Roy. Soc. London. Ser. A. 181, 411–415 (1943). [MF 8852]

This is a discussion concerning K. Fuchs' paper in the same Proc. 179, 340–361 (1942); these Rev. 4, 29.

Jaffé, George. A statistical theory of liquids. II. Phys. Rev. (2) 63, 313–321 (1943). [MF 8240]

The vapor pressure law $n_2=n_1\pi_2/[\pi_1+(\omega S/V_1)]$ was deduced in part I [same Rev. (2) 62, 463–476 (1942); these Rev. 4, 208] under the restriction that the radii of curvature of the liquid boundary $\gg$ range of molecular action, so that the surface term is insignificant, and, with the aid of the expressions $\log\pi_1=-u_1/kT$, $\log\pi_2=-u_2/kT$, the equation may be written in the simple form

$$n_2/n_1=\exp[(u_2-u_1)/kT],$$

where $u_1=2\pi n_1\psi(\sigma_1)$, $u_2=2\pi n_2\psi(\sigma_2)$. It is now assumed that the density ρ_2 in the vapor phase is not definitely less than ρ_1, the density in the liquid phase. Molar quantities are introduced by writing $U_1^*=N_0u_1$, $U_2^*=N_0u_2$, where N_0 is Avogadro's number. The molar heat of vaporization L (which still contains the external work) is then

$$L^*=U_2^*-U_1^*+Mp(v_2-v_1),$$

where M denotes the molecular weight and v_1, v_2 specific volumes. If it is assumed that the molecular constitution is the same in the two phases,

$$n_2/n_1=\rho_2/\rho_1=\exp\{[Mp(v_2-v_1)-L]/RT\}.$$

It is pointed out that this and the first relation might have been obtained directly by means of Boltzmann's theorem.

Similarly, by neglecting S in the formula

$$\gamma=-kTn_1(\omega/\pi_1)/(1+\omega S/\pi_1V_1),$$

a simple formula is found for the surface tension

$$\gamma=kTn_1\int_0^\infty[1-\exp(u_1-u)/kT]dh.$$

Since $u-u_1$ is always positive γ is positive. A first approximation is obtained by assuming the molecular density n to be constant even in the surface layer. Then

$$u=\pi n_1\{2\psi(\sigma_1)-\psi(h)+h\chi(h)\},\qquad h\geqq\sigma_1,$$
$$u=\pi n\{\psi(\sigma_1)+h\chi(\sigma_1)\},\qquad h\leqq\sigma_1,$$

and γ is expressed in the form $\gamma=kTn\sigma^*$, where σ^*, the average thickness of the surface layer, is of the form $\sigma^*=z\sigma_1$, where z is the sum of two integrals. The surface layer may be regarded as a separate phase with a surface density $\nu_1=n_1\sigma^*$ and an equation of state $\gamma=\nu_1kT$. The history of this idea is given and its relation to the law of Eötvös is pointed out. After a comparison between theoretical and empirical data the law of Eötvös is discussed in more detail. Formulae valid near the critical point are obtained by abandoning the assumption in the previous derivation of the potential for the surface layer that the contribution on the vapor side may be neglected. *H. Bateman.*

AUTHOR INDEX

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